

Upward Adequacy Theorem
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In proving DAT, we proved that for any set of FOL sentences, Σ , if Σ is consistent, then any completed tree for it, $T[\Sigma]$, will be open. But what about when Σ is inconsistent? Here is the theorem we need to prove:

(UAT) For any set Σ , if Σ is inconsistent, then no $T[\Sigma]$ will contain any open branches.

This, it turns out, is difficult to prove as stated. Fortunately, we can use some logical transformation techniques (contraposition) to restate this theorem in a more easily provable form.

(UAT*) For any set of FOL sentences, Σ , if some $T[\Sigma]$ is open, then Σ is consistent.

So, we want to prove that all the sentences in any Σ in the trunk of an open completed tree are true on some I. To prove this, we'll start with the sentences at the **bottom** of an open branch in the open tree and work up to the sentences in Σ . At the bottom of any open completed tree are literals. So, these will have to be our base. But here is our first problem.

We could define a complexity function, $COMP[\phi]$, as the number of operators in ' ϕ '. But now consider the complexity of literals. There are two different possible measures. For example, $COMP[P(a)] = 0$, and $COMP[\neg Q(b)] = 1$. This wouldn't be a problem except for the fact that many non-literals have the same COMP measure as some literals. For example, $COMP[\forall x Fx] = 1$ and $COMP[(P(a) \rightarrow Q(b))]$ = 1. This means that complexity won't be a good way of "numericalizing" our sentences for a proof by mathematical induction. We need a different way to partition sentences. We want literals to have a measure of zero, so they can serve as the base for our inductive proof. Moreover, we want the increasing measure of sentences to be the inverse of the rules. That is, if we have a sentence, ' ϕ ', and $MEASURE[\phi] = n$, then if we apply any of rules to ' ϕ ', the resulting sentence, call it ' ψ ', should have a lower measure, i.e., we want $MEASURE[\psi] = n - 1$. If we can find such a way to invoke the natural numbers in our measurement of sentences, we'll be able to prove UAT* (and thus UAT) by mathematical induction.

Consider the following. Let the *generating predecessor* or *immediate predecessor* of a sentence on a branch be the sentence(s) to which a rule is applied to obtain the resulting sentence. Call the resulting sentence the *immediate descendant*. If ' ϕ ' is the immediate descendant of ' ψ ', and ' χ ' is the immediate descendant of ' ϕ ', then ' χ ' is a *descendant* of ' ψ '. For any collection of descendants on a branch, there is one *original ancestor collection* for that collection of descendants. The original ancestor collection is always some subset of Σ .

We should be able to use the notions of predecessors, descendants and ancestors to define a measure that gives literals a measure of zero, and, for each non-literal sentence, its measure is a function of how close (or far) the sentence is to (or from) being a literal.

All members of the collection of descendants on a branch descend from an original ancestor collection. To get from a predecessor to a literal requires the application of a rule. And, with each application of a rule, the resulting descendant is *closer* to being a literal. Hence, we can define a measurement function, $AGE_B[\phi]$, as the number of applications of rules it would take to process a sentence of FOL, ' ϕ ', through all of its descendants on a branch, B, all the way to literals.

For example, consider " $((P(a) \rightarrow (Q(b) \rightarrow R(c))) \wedge S(d))$ ". (Call it "Baxter" for convenience.) There is a sense in which we can get Baxter to have a literal descendant with one application.

$$\begin{array}{c} ((P(a) \rightarrow (Q(b) \rightarrow R(c))) \wedge S(d))\checkmark \\ | \\ (P(a) \rightarrow (Q(b) \rightarrow R(c))) \\ S(d) \end{array}$$

This took only the application of the Conjunction Rule to get a single literal descendant. However, at that same level, one of the descendants of Baxter is not a literal. So, we would need to process it as well, in this case by applying the Conditional Rule.

$$\begin{array}{c} ((P(a) \rightarrow (Q(b) \rightarrow R(c))) \wedge S(d))\checkmark \\ | \\ (P(a) \rightarrow (Q(b) \rightarrow R(c)))\checkmark \\ S(d) \\ / \quad \backslash \\ \neg P(a) \quad (Q(b) \rightarrow R(c)) \end{array}$$

Now, are all the descendants on a (completed) branch literal? Yes—on the left branch.

So, it looks as if the $AGE_B[((P(a) \rightarrow (Q(b) \rightarrow R(c))) \wedge S(d))]$ = 2

But wait, what about " $(Q(b) \rightarrow R(c))$ " on the right branch? You are probably thinking, "That's not a literal!" You are correct, that isn't a literal. But it is on another branch. That's what the subscript "B" was doing in the AGE-function. So, there will often be different AGEs for sentences depending on the branches considered. We will normally gloss over this distinction. But to make sure this works, let's explicitly number branches from left to right. That is, the leftmost branch will be known as " B_1 ". The next branch to the right of B_1 is called " B_2 ", etc. For any given set of sentences, then, we will be able to construct and label its branches in such a way that for any branch we would be able to determine the AGE of any sentence on that branch.

For a specific example, here is the completed tree from above:

$$\begin{array}{c} ((P(a) \rightarrow (Q(b) \rightarrow R(c))) \wedge S(d))\checkmark \\ | \\ (P(a) \rightarrow (Q(b) \rightarrow R(c)))\checkmark \\ S(d) \\ / \quad \backslash \\ \neg P(a) \quad (Q(b) \rightarrow R(c))\checkmark \\ \quad \quad / \quad \backslash \\ \quad \quad \neg Q(b) \quad R(c) \end{array}$$

Now we can label these branches “B₁”, “B₂”, and “B₃”, respectively.

The result is that AGE_{B₁}[Baxter] = 2, AGE_{B₂}[Baxter] = 3, and AGE_{B₃}[Baxter] = 3.

We could simplify things here and say that the overall AGE of any sentence is whatever AGE is oldest. But we don’t need this simplification. Moreover, since we are going to be talking about branches in the abstract, we can dispense with the subscripts. [Note: we will also, below, consider the AGE of a *set* of sentences, Γ . A set of sentences, Γ , having a particular AGE, say, k , does not mean that every member of Γ is individually of AGE k . Some might have AGE = 0. But the oldest member of an AGE k set, Γ , will be of AGE k .]

Now, we need just one more thing before we can prove (UAT*). Remember Mr. T?



Well, not that one. For us, Mr. T is an interpretation that makes all of the literals on a branch true. [Mr. T is what the OBI method generated for each open branch we applied it to.] We are going to ask for Mr. T’s help in proving the Upward Adequacy Theorem. So, first we need to prove a lemma about Mr. T. We need to prove that *some* Mr. T exists for any open tree (indeed, for any completed open branch).

Mr. T Existence Lemma

(MTEL) If a branch is open, then there is an interpretation () that makes all its literals true.

Proof:

By the definition of branch closure, if a branch is open, then it does not contain an atom, ‘ ϕ ’, and its negation, ‘ $\neg\phi$ ’. Since Mr. T () is supposed to make all the literals on the branch true, then, because the set of literals on the branch does not include both ‘ ϕ ’ and ‘ $\neg\phi$ ’, this is possible. Thus, Mr. T () always exists, that is, for any open branch, there is *some* Mr. T () that makes all the literals on the branch true.

Proof of UAT* for FOL

For any set of sentences, Σ , if some $T[\Sigma]$ is open, then Σ is consistent.

Proof by conditional proof with embedded mathematical induction on the AGE of sentences
 For conditional proof, we will assume that some $T[\Sigma]$ is open and use this to show that Σ is consistent. We will do this by showing that Mr. T () makes all the literals on an open branch of $T[\Sigma]$ true. Then we will assume that Mr. T () makes true all the sentences on the open branch, B, of all AGEs, n, up to and including level k. From this we will show that Mr. T () makes true all the sentences of AGE k + 1 on B. Finally, by mathematical induction, we will conclude that Mr. T () makes all sentences of any AGE on B true, and so (since the sentences in Σ are on every branch of the tree) makes all the sentences in Σ true. This means that Σ is consistent. [Note: this proof does not “ladder” up through the levels of the open branch, B, since, for each n, AGE n sentences can occur at almost *any* branch-level and at *multiple* branch-levels.]

Induction Set: The sentences on an open branch of $T[\Sigma]$ (where $T[\Sigma]$ is a completed tree for Σ).

Partition: in terms of AGE[ϕ], i.e., the AGE of each sentence, ‘ ϕ ’, on that open branch.

Basis: All ‘ ϕ ’ on B, such that AGE[ϕ] = 0, meaning everything on B (wherever it occurs) that is a literal. Since, by hypothesis, B is an open branch of $T[\Sigma]$, then, by the MTEL, we can be sure that there is an interpretation, Mr. T () that makes all the literals on B true. So, all ‘ ϕ ’ on B such that AGE[ϕ] = 0 are true on Mr. T ()—no matter where they occur on B (including in Σ itself).

Induction Step:

Inductive Hypothesis (IH): Mr. T () makes every sentence ‘ ϕ ’ true, where AGE[ϕ] = n $\leq$ k [and thus makes every member of any AGE k set of sentences, Γ , true].

Show: Assuming IH, Mr. T () makes all sentences ‘ ψ ’ true where AGE[ψ] = k + 1.

We will show this by cases.

Every case will have the following form: ‘ ϕ ’ (or sometimes a set of sentences, Γ) is the immediate descendant of some ‘ ψ ’ on B, and AGE[ϕ/Γ] = k. This means that AGE[ψ] = k + 1. Moreover, ‘ ϕ/Γ ’ is covered by the IH, i.e., Mr. T makes ‘ ϕ ’/everything in Γ true. For each tree rule application that gets ‘ ϕ/Γ ’ from ‘ ψ ’, we will show that Mr. T also makes ‘ ψ ’ true. By the ladder principle, then, we can be sure that Mr. T makes every sentence, of any AGE, on B true (including all those in Σ , the original ancestor collection for all descendants on B).

Case 1: ‘ ϕ ’ is the AGE k immediate descendant of ‘ ψ ’ on B via the ***Double Negation Rule***.

So, ‘ ψ ’ = ‘ $\neg\neg\phi$ ’. By IH, Mr. T makes ‘ ϕ ’ true. So, Mr. T makes ‘ $\neg\phi$ ’ false [TC $\neg$]. So, Mr. T makes ‘ $\neg\neg\phi$ ’ true [TC $\neg$]. And that is what we wanted to show (namely, that if Mr. T makes ‘ ϕ ’ true, then Mr. T makes ‘ ψ ’ true as well).

Case 2: Γ is the AGE k set immediate descendant of ' ψ ' by application of the **Conjunction Rule**. So, ' ψ ' = ' $(\chi \wedge \tau)$ ' and $\Gamma = \{\chi, \tau\}$. Since, by IH, Mr. T makes everything in Γ true, it makes ' χ ' true and makes ' τ ' true. But this means Mr. T makes ' $(\chi \wedge \tau)$ ' true as well [TC $\wedge$]. So, if Mr. T makes everything in Γ true, then it makes ' ψ ' true.

Case 3: ' ϕ ' is the AGE k immediate descendant of ' ψ ' by use of the **Negated Conjunction Rule**. So, ' ψ ' = ' $\neg(\chi \wedge \tau)$ ' and ' ϕ ' is either ' $\neg\chi$ ' or ' $\neg\tau$ '.

Subcase 3a: ' ϕ ' is ' $\neg\chi$ '. Since (by IH) Mr. T makes ' ϕ ' true, it makes ' χ ' false [TC $\neg$]. But that means Mr. T makes ' $(\chi \wedge \tau)$ ' false [TC $\wedge$]. So, Mr. T makes ' $\neg(\chi \wedge \tau)$ ' true [TC $\neg$], and thus makes ' ψ ' true.

Subcase 3b: ' ϕ ' is ' $\neg\tau$ '. Since (by IH) Mr. T makes ' ϕ ' true, it makes ' τ ' false [TC $\neg$]. But that means Mr. T makes ' $(\chi \wedge \tau)$ ' false [TC $\wedge$]. So, Mr. T makes ' $\neg(\chi \wedge \tau)$ ' true [TC $\neg$], and thus makes ' ψ ' true.

Since those are the only two subcases, if Mr. T makes ' ϕ ' true, then it makes ' ψ ' true too.

Case 4: ' ϕ ' is the AGE k immediate descendant of ' ψ ' by appeal to the **Disjunction Rule**. So, ' ψ ' = ' $(\chi \vee \tau)$ ' and ' ϕ ' is either ' χ ' or ' τ '.

Subcase 4a: ' ϕ ' is ' χ '. Since (by IH) Mr. T makes ' ϕ ' true, it makes ' χ ' true. So, Mr. T makes ' $(\chi \vee \tau)$ ' true [TC $\vee$], and thus makes ' ψ ' true.

Subcase 4b: ' ϕ ' is ' τ '. Since (by IH) Mr. T makes ' ϕ ' true, it makes ' τ ' true. So, Mr. T makes ' $(\chi \vee \tau)$ ' true [TC $\vee$], and thus makes ' ψ ' true.

So, if Mr. T makes ' ϕ ' true, then it makes ' ψ ' true.

Case 5: Γ is the AGE k set immediate descendant of ' ψ ' by appeal to the **Negated Disjunction Rule**. So, ' ψ ' = ' $\neg(\chi \vee \tau)$ ' and $\Gamma = \{\neg\chi, \neg\tau\}$. [PROOF LEFT FOR HOMEWORK—see Case 2]

Case 6: ' ϕ ' is the AGE k immediate descendant of ' ψ ' by appeal to the **Conditional Rule**. So, ' ψ ' = ' $(\chi \rightarrow \tau)$ ' and ' ϕ ' is either ' $\neg\chi$ ' or ' τ '.

Subcase 6a: ' ϕ ' is ' $\neg\chi$ '. Since Mr. T makes ' ϕ ' true, it makes ' χ ' false [TC $\neg$]. So, Mr. T makes ' $(\chi \rightarrow \tau)$ ' true [TC $\rightarrow$], so it makes ' ψ ' true.

Subcase 6b: ' ϕ ' is ' τ '. Since Mr. T makes ' ϕ ' true, it makes ' τ ' true. So, Mr. T makes ' $(\chi \rightarrow \tau)$ ' true [TC $\rightarrow$], and thus makes ' ψ ' true.

So, if Mr. T makes ' ϕ ' true, then it makes ' ψ ' true.

Case 7: Γ is the AGE k set immediate descendant of ' ψ ' by appeal to the **Negated Conditional Rule**. So ' ψ ' = ' $\neg(\chi \rightarrow \tau)$ ' and $\Gamma = \{\chi, \neg\tau\}$. Since (by IH) Mr. T makes everything in Γ true, it makes ' χ ' true and ' $\neg\tau$ ' true. Thus, Mr. T makes ' τ ' false [TC $\neg$] and so makes ' $(\chi \rightarrow \tau)$ ' false [TC $\rightarrow$], thereby making ' $\neg(\chi \rightarrow \tau)$ ' true [TC $\neg$]. Hence, Mr. T makes ' ψ ' true. So, if Mr. T makes everything in Γ true, then it makes ' ψ ' true.

Case 8: Γ is the AGE k set immediate descendant of ' ψ ' by appeal to the **Biconditional Rule**. So ' ψ ' = ' $(\chi \leftrightarrow \tau)$ ' and $\Gamma = \{\chi, \tau\}$ or $\Gamma = \{\neg\chi, \neg\tau\}$.

Subcase 8a: Γ is $\{\chi, \tau\}$. Since (by IH) Mr. T makes everything in Γ true, it makes ' χ ' true and makes ' τ ' true. So, Mr. T makes ' $(\chi \leftrightarrow \tau)$ ' true [TC $\leftrightarrow$], thereby making ' ψ ' true.

Subcase 8b: Γ is $\{\neg\chi, \neg\tau\}$. Since (by IH) Mr. T makes everything in Γ true, it makes ' $\neg\chi$ ' true and makes ' $\neg\tau$ ' true. Thus, Mr. T makes ' χ ' false and makes ' τ ' false [TC $\neg$]. So, Mr. T makes ' $(\chi \leftrightarrow \tau)$ ' true [TC $\leftrightarrow$], thereby making ' ψ ' true.

So, if Mr. T makes everything in Γ true, then it makes ' ψ ' true.

Case 9: Γ is the AGE k set immediate descendant of ' ψ ' via the **Negated Biconditional Rule**. So ' ψ ' = ' $\neg(\chi \leftrightarrow \tau)$ ' and $\Gamma = \{\chi, \neg\tau\}$ or $\Gamma = \{\neg\chi, \tau\}$. [PROOF LEFT FOR HOMEWORK]

Case 10: ' ϕ ' is the AGE k immediate descendant of ' ψ ' by appeal to the **Universal Quantifier Rule**. So ' ψ ' = ' $\forall x\theta_{(x)}$ ' and ' ϕ ' = ' $\theta_{(m)}$ '. To understand this case, we need to think about what it means for Mr. T to make true all of the sentences on B of all AGEs, $n \leq k$. First, notice that $\text{AGE}[\theta_{(m)}] = k$. Consider any $\alpha \in D$ for the interpretation. Notice that for any name ' i ' that designates α on Mr. T, any sentence ' $\theta_{(i)}$ ' on B is such that $\text{AGE}[\theta_{(i)}] = k$. Now, we know that there isn't a sentence ' $\neg\theta_{(i)}$ ' on B because B is open, and the tree is completed and ' $\forall x\theta_{(x)}$ ' is on the branch. Hence, if ' i ' occurred in any sentence on B, then the Universal Quantifier Rule requires that we put ' $\theta_{(i)}$ ' on B. So, if ' $\neg\theta_{(i)}$ ' were anywhere on B, the branch would be closed, contra our supposition. Think about what this means. For any name ' i ' we might pick, ' $\theta_{(i)}$ ' is on the branch, and $\text{AGE}[\theta_{(i)}] = k$. Hence, by IH, Mr. T makes ' $\theta_{(i)}$ ' true for any ' i ' we might pick, no matter what $\alpha \in D$ Mr. T makes ' i ' designate. But that means all $\alpha \in D$ satisfy ' $\theta_{(x)}$ ' on Mr. T [Def Sat]. Hence, Mr. T makes ' $\forall x\theta_{(x)}$ ' true [TC $\forall$] and thereby makes ' ψ ' true. Thus, if Mr. T makes ' ϕ ' true, then it makes ' ψ ' true.

Case 11: ' ϕ ' is the AGE k immediate descendant of ' ψ ' by appeal to the **Negated Universal Rule**. So, ' ψ ' = ' $\neg\forall x\theta_{(x)}$ ' and ' ϕ ' = ' $\neg\theta_{(m)}$ '. [PROOF LEFT FOR HOMEWORK—see Case 12]

Case 12: ' ϕ ' is the AGE k immediate descendant of ' ψ ' by appeal to the **Existential Quantifier Rule**. So, ' ψ ' = ' $\exists x\theta_{(x)}$ ' and ' ϕ ' = ' $\theta_{(m)}$ '. Once again, notice that $\text{AGE}[\theta_{(m)}] = k$. Since, by IH, Mr. T makes ' $\theta_{(m)}$ ' true, there is some $\alpha \in D$ that ' m ' designates on Mr. T, such that if we replaces at least one occurrence of ' m ' in ' $\theta_{(m)}$ ' with an " x " (to form ' $\theta_{(x)}$ '), then α would satisfy this open wff [Def Sat]. But since some $\alpha \in D$ satisfies ' $\theta_{(x)}$ ' on Mr. T, Mr. T makes ' $\exists x\theta_{(x)}$ ' true [TC $\exists$], and thus makes ' ψ ' true. So, if Mr. T makes ' ϕ ' true, then it makes ' ψ ' true.

Case 13: ‘ ϕ ’ is the AGE k immediate descendant of ‘ ψ ’ by appeal to the *Negated Existential Rule*. So ‘ ψ ’ = ‘ $\neg \exists x \theta_{(x)}$ ’ and ‘ ϕ ’ = ‘ $\neg \theta_{(m)}$ ’. [PROOF LEFT FOR HOMEWORK—see Case 10]

Case 14: ‘ ϕ ’ is the AGE k immediate descendant of something by appeal to the *Identity Rule*. Here we have a different situation, since instead of an AGE $k + 1$ level sentence, ‘ ψ ’, in this case we have something like an AGE $k + 1$ level set $\Gamma = \{(\mathbf{m} = \mathbf{i}), \theta_{(m)}\}$ with ‘ ϕ ’ = ‘ $\theta_{(i)}$ ’. What should we say in this case?

First, note that ‘ $(\mathbf{m} = \mathbf{i})$ ’ is a literal. Hence, Mr. T makes it true by our IH, since $\text{AGE}[(\mathbf{m} = \mathbf{i})] = 0$ and 0 must be $\leq k$. [Note that the fact that 0 could be strictly $< k$ is why we have to do strong mathematical induction in proving UAT.] What is strange about this case, however, is that the *Identity Rule* requires *two* elements at the start. So, our AGE function doesn’t seem to apply. Second, assuming that $\text{AGE}[\theta_{(m)}] = k$, then $\text{AGE}[\theta_{(i)}] = k$ too. So, again, this rule is defying our AGE function. One “fix” would be to declare an exemption to the AGE function in order to make this tree rule fit. One possible exemption is to add one to the AGE of any wff that is used in the *Identity Rule*. This would make $\text{AGE}[\theta_{(m)}] = k + 1$ (which is what we want). However, this is arbitrary and *ad hoc*. So, we shouldn’t use it. Instead, let’s look at Γ again. Remember that what we want to prove is that if Mr. T makes ‘ ϕ ’ true, then it makes everything in Γ true. But in this case, this isn’t one sentence; it is two. But what this means is that it is better to think of Γ as an implicit conjunction, ‘ ψ ’ = ‘ $((\mathbf{m} = \mathbf{i}) \wedge \theta_{(m)})$ ’, in this case. If we do, then since $\text{AGE}[\theta_{(m)}] = \text{AGE}[\theta_{(i)}] = k$, and $\text{AGE}[(\mathbf{m} = \mathbf{i})] = 0$, it follows that, with the “ $\wedge$ ” placed between them adding a processing step to get to literals, $\text{AGE}[(\mathbf{m} = \mathbf{i}) \wedge \theta_{(m)}] = k + 1$. That is, $\text{AGE}[\psi] = k + 1$!

Now, there is a cheap way to argue that, if Mr. T makes ‘ ϕ ’ true, then it makes ‘ ψ ’ true too (viz., by pointing out that the conjuncts of ‘ ψ ’ are equal in AGE to or younger than ‘ ϕ ’, which would mean that Mr. T makes them true by fiat). Instead, let’s show that given that Mr. T makes ‘ ϕ ’ (i.e., ‘ $\theta_{(i)}$ ’) true and makes ‘ $(\mathbf{m} = \mathbf{i})$ ’ true (because it is a literal), it follows that Mr. T makes ‘ $\theta_{(m)}$ ’ true, and thereby makes ‘ ψ ’ true. It should be clear that if Mr. T makes ‘ $(\mathbf{m} = \mathbf{i})$ ’ true, then whatever $\alpha \in D$ Mr. T makes ‘ $\mathbf{i}$ ’ designate, Mr. T has ‘ $\mathbf{m}$ ’ designate that α as well [TC =]. But since α satisfies ‘ $\theta_{(x)}$ ’ on Mr. T (since Mr. T makes ‘ $\theta_{(i)}$ ’ true [Def Sat]), that would mean that Mr. T makes ‘ $\theta_{(m)}$ ’ true as well. But now, since Mr. T makes ‘ $(\mathbf{m} = \mathbf{i})$ ’ true and makes ‘ $\theta_{(m)}$ ’ true, it follows that Mr. T makes ‘ $((\mathbf{m} = \mathbf{i}) \wedge \theta_{(m)})$ ’ true [TC $\wedge$], thereby making ‘ ψ ’ true. Thus, if Mr. T makes ‘ ϕ ’ true, then it makes ‘ ψ ’ true (and hence makes everything in Γ true).

Case 15*: ‘ ϕ ’ is *supposedly* the AGE k immediate descendant of ‘ ψ ’ by appeal to the *Negated Identity Rule*. That would mean ‘ ψ ’ is ‘ $(\mathbf{m} \neq \mathbf{m})$ ’. But this ‘ ψ ’ cannot be on B, since then the branch would be closed, contra our assumption. So, there is no ‘ ϕ ’ that is the immediate descendant of any ‘ ψ ’ by appeal to this rule. Since this case is impossible, it does not factor into our conclusion.

Assuming that we can finish the cases that were left for you to do (Cases 5, 9, 11, 13), we will have established that, on the assumption of our IH, Mr. T makes every AGE $k + 1$ sentences on an open branch, B, true. Discharging the assumption of IH, then, this lets us formulate the ladder principle for our mathematical induction.

Ladder Principle: For any sentence ‘ ϕ ’, if Mr. T makes ‘ ϕ ’ true when $\text{AGE}[\phi] = n \leq k$ [thereby making all the members of any AGE k set of sentences, Γ , true], then Mr. T makes true any ‘ ψ ’ that is the immediate predecessor of ‘ ϕ ’ [or Γ] (making $\text{AGE}[\psi] = k + 1$).

Conclusion: Since Mr. T makes any sentence ‘ ϕ ’ on B true where $\text{AGE}[\phi] = 0$ [and thus every member of any set of sentences, Γ , of AGE 0], by the Ladder Principle, it follows that for any sentence ‘ ϕ ’ on B where $\text{AGE}[\phi] = 1$, Mr. T makes that sentence true. Since we now have that Mr. T makes true all AGE 0 and all AGE 1 sentences on B [and so every member of any set of sentences, Γ , of AGE 1], LP gives us that Mr. T makes true all ‘ ϕ ’ on B where $\text{AGE}[\phi] = 2$. And so on, for all sentences on B of all AGES. So, Mr. T makes all the sentences on B true, meaning it makes all the sentences in Σ true! In other words, Σ is consistent. [Notices how we establish that all the sentences in Σ are true on Mr. T basically by overshooting—that is, by establishing that Mr. T makes all the sentences on B (which include all those in Σ) true.]

Thus, we have shown:

(UAT*) For any set of FOL sentences, Σ , if some $T[\Sigma]$ is open, then Σ is consistent.

Combining DAT and UAT:

Putting them together, we get that, for any Σ , if it is consistent, then all of its trees are open, and if at least one of its trees is open, then it is consistent.

We’d like this in a snappier “if and only if” form, but, as it stands, the first part of it is about *all* $T[\Sigma]$ and the second part is about *some* $T[\Sigma]$. What can we do? Well, consider what would follow if some $T[\Sigma]$ were open, but some *other* $T[\Sigma]$ were closed. By UAT, the first tree would indicate that Σ is consistent. But, by (the contrapositive of) DAT (i.e., if some $T[\Sigma]$ is closed, then Σ is inconsistent), the second tree would indicate that Σ is inconsistent. But that’s a contradiction, so for any Σ , if *some* $T[\Sigma]$ is open, then *all* $T[\Sigma]$ are open (and if *some* $T[\Sigma]$ is closed, then *all* $T[\Sigma]$ are closed). So, we can restate UAT thus: for any Σ , if *all* $T[\Sigma]$ are open, then Σ is consistent.

But now we can combine UAT with DAT (D/UAT):

For any (finite) set of FOL sentences, Σ , Σ is consistent if and only if all of its trees are open.

Note well that this covers inconsistency and closed trees by logical extension (negating both sides of the biconditional). Thus, our deductive apparatus (the Tree Rules) is fully semantically reliable. Applying it to any set of sentences of FOL will never lead us astray, regarding the set’s semantic properties, and the semantic properties of every set can be determined by our system.