

0. The Basics

Double-quotes indicate mention of symbols; single-quotes function as permeable corner/Quine quotes. Let the symbols “ α ” and “ β ” (and “ γ ”, “ δ ”, etc.) be *metalinguage variables* that range over objects from the domain of discourse, D, specified by some interpretation, I. Let the symbols “ $\mathfrak{m}$ ” and “ $\mathfrak{n}$ ” be *metalinguage variables* that can be filled in (i.e., replaced) with any individual constant (name) from the object-language, FOL, (even inside single quotation marks). Let the symbols “ ϕ ” and “ ψ ” be *metalinguage variables* that can be filled in (i.e., replaced) with wffs of FOL (even inside single quotation marks). Let “ $\mathcal{F}$ ”, “ $\mathcal{G}$ ”, “ $\mathcal{H}$ ”, etc. be *metalinguage variables* that can be filled in with any predicate from FOL (even inside single quotation marks).

I. Atomic Sentences [TC Atomic]

- ‘ $\mathcal{F}(\mathfrak{n})$ ’ is true on I iff for some $\alpha \in D$, I designates α by ‘ $\mathfrak{n}$ ’ and the object α is in the set that is the extension of the predicate, ‘ $\mathcal{F}$ ’, i.e., $\alpha \in \mathcal{F}$. ‘ $\mathcal{F}(\mathfrak{n})$ ’ is false on I iff ... and $\alpha \notin \mathcal{F}$.
 [Example: “J(a)” is true on I iff for some $\alpha \in D$, I designates α by “a” and $\alpha \in \text{“J”}$. “J(a)” is false on I iff for some $\alpha \in D$, I designates α by “a” and $\alpha \notin \text{“J”}$.]
- ‘ $\mathcal{H}(\mathfrak{n}, \mathfrak{m})$ ’ is true on I iff for some $\alpha \in D$ and some $\beta \in D$ (where α and β may be the same object), I designates α by ‘ $\mathfrak{n}$ ’ and β by ‘ $\mathfrak{m}$ ’ (where ‘ $\mathfrak{n}$ ’ and ‘ $\mathfrak{m}$ ’ may be the same name, if α and β are the same object) and the *ordered pair* $\langle \alpha, \beta \rangle \in \mathcal{H}$. ‘ $\mathcal{H}(\mathfrak{n}, \mathfrak{m})$ ’ is false on I iff for some $\alpha \in D$ and some $\beta \in D$, I designates α by ‘ $\mathfrak{n}$ ’ and β by ‘ $\mathfrak{m}$ ’ and $\langle \alpha, \beta \rangle \notin \mathcal{H}$.
 [Example: “G(a,b)” is true on I iff for some $\alpha \in D$ and some $\beta \in D$ (where α and β may be the same object), I designates α by “a” and β by “b” and $\langle \alpha, \beta \rangle \in \text{“G”}$. “G(a,b)” is false on I iff for some $\alpha \in D$ and some $\beta \in D$ (where α and β may be the same object), I designates α by “a” and β by “b” and $\langle \alpha, \beta \rangle \notin \text{“G”}$.]

Etc. for 3-place predicates, with sets of *ordered triples* as their extensions, and so on.

II. The Sentential Connectives

In the following, “ ϕ ” and “ ψ ” are variables for *sentences* (not open wffs) of FOL.

- [TC $\neg$] ‘ $\neg \phi$ ’ is true on I iff ‘ ϕ ’ is false on I. ‘ $\neg \phi$ ’ is false on I iff ‘ ϕ ’ is true on I.
- [TC $\wedge$] ‘ $(\phi \wedge \psi)$ ’ is true on I iff BOTH ‘ ϕ ’ is true on I AND ‘ ψ ’ is true on I. ‘ $(\phi \wedge \psi)$ ’ is false on I iff EITHER ‘ ϕ ’ is false on I OR ‘ ψ ’ is false on I (or both).
- [TC $\vee$] ‘ $(\phi \vee \psi)$ ’ is true on I iff EITHER ‘ ϕ ’ is true on I OR ‘ ψ ’ is true on I (or both). ‘ $(\phi \vee \psi)$ ’ is false on I iff BOTH ‘ ϕ ’ is false on I AND ‘ ψ ’ is false on I.
- [TC $\rightarrow$] ‘ $(\phi \rightarrow \psi)$ ’ is true on I iff EITHER ‘ ϕ ’ is false on I OR ‘ ψ ’ is true on I (or both). ‘ $(\phi \rightarrow \psi)$ ’ is false on I iff BOTH ‘ ϕ ’ is true on I AND ‘ ψ ’ is false on I.
- [TC $\leftrightarrow$] ‘ $(\phi \leftrightarrow \psi)$ ’ is true on I iff EITHER both ‘ ϕ ’ and ‘ ψ ’ are true on I OR both ‘ ϕ ’ and ‘ ψ ’ are false on I. ‘ $(\phi \leftrightarrow \psi)$ ’ is false on I iff EITHER both ‘ ϕ ’ is false and ‘ ψ ’ is true on I OR both ‘ ϕ ’ is true and ‘ ψ ’ is false on I.

III. The Quantifiers

- [TC $\exists$] ' $\exists x\phi$ ' is true on I iff some $\alpha \in D$ satisfies ' ϕ ' on I. ' $\exists x\phi$ ' is false on I iff every $\alpha \in D$ does not satisfy ' ϕ ' on I.
- [TC $\forall$] ' $\forall x\phi$ ' is true on I iff every $\alpha \in D$ satisfies ' ϕ ' on I. ' $\forall x\phi$ ' is false on I iff some $\alpha \in D$ does not satisfy ' ϕ ' on I.
- $\alpha \in D$ satisfies the open wff ' ϕ ' on I iff substituting a (perhaps arbitrarily introduced) name ' $\mathfrak{u}$ ', that designates α on I, for every occurrence of the open variable in ' ϕ ', results in a sentence that is true on I. $\alpha \in D$ does not satisfy ' ϕ ' on I iff the resulting sentence is false on I. [Call this the *Definition of Satisfaction* or "Def. Sat."] [Satisfaction of complex open wffs is governed by the truth-conditions of the operators.]

IV. Identity

- [TC =] Any identity claim, ' $(\mathfrak{m} = \mathfrak{u})$ ', is true on I iff for some $\alpha \in D$ I designates α by ' $\mathfrak{m}$ ' and also by ' $\mathfrak{u}$ '. Any identity claim, ' $(\mathfrak{m} = \mathfrak{u})$ ', is false on I iff there is no $\alpha \in D$ such that I designates α by ' $\mathfrak{m}$ ' and also by ' $\mathfrak{u}$ '. This means there is an $\alpha \in D$ and an $\beta \in D$, where I designates α by ' $\mathfrak{m}$ ' and β by ' $\mathfrak{u}$ ', and α and β are distinct objects of D. So, if ' $(\mathfrak{m} = \mathfrak{u})$ ' is false on I, then, if ' $\mathfrak{m}$ ' designates α on I and ' $\mathfrak{u}$ ' designates β on I, $\langle \alpha, \beta \rangle \notin "="$ on I and $\langle \beta, \alpha \rangle \notin "="$ on I.
[Example: " $(a = b)$ " is true on I iff for some $\alpha \in D$, both "a" and "b" designate α on I. " $(a = b)$ " is false on I iff there is no $\alpha \in D$ such that "a" and "b" both designate α on I. So, there is an $\alpha \in D$ and a $\beta \in D$, where "a" designates α on I and "b" designates β on I, and α and β are distinct objects of D, meaning that $\langle \alpha, \beta \rangle \notin "="$ on I and $\langle \beta, \alpha \rangle \notin "="$ on I.]

NOTE: For every possible interpretation, I, the identity predicate (the "=") has the following extension: I makes every $\alpha \in D$, $\langle \alpha, \alpha \rangle \in "="$. Call this the *Definition of Identity* or "Def. Id."

V. Substitution Rules

- If α and β are the same object, then you can substitute the *metalinguage variable* " α " for any use of the *metalinguage variable* " β " in your proof, and vice versa.
- For any interpretation, I, if ' $(\mathfrak{m} = \mathfrak{u})$ ' is true on I, then you can substitute the *metalinguage variable* used to pick out the object that I designates by ' $\mathfrak{m}$ ' (e.g., the variable " α ") for any *metalinguage variable* used to pick out the object that I designates by ' $\mathfrak{u}$ ' (e.g., the variable " β "), and vice versa.

VI. Example Semantic Proofs.

1. Show that " $(a = b)$ " is contingent.

Proof: It is contingent iff some I_1 makes it true AND some I_2 makes it false. Since " $(a = b)$ " is true on I_1 , there is some $\alpha \in D$, such that I_1 designates α by "a" and designates α by "b" [TC =]. So let I_1 have $D = \{1\}$, "a" designates 1, "b" designates 1.

Since " $(a = b)$ " is false on I_2 , for some $\alpha \in D$, I_2 designates α by "a" and for some $\beta \in D$, I_2 designates β by "b", and α and β are distinct [TC =]. So let I_2 have $D = \{1, 2\}$, "a" designates 1, "b" designates 2. Since I_1 makes it true and I_2 makes it false, the sentence is contingent.

2. Show that “(a = a)” is L-true.

Proof: Suppose not, so there is some interpretation, call it I, that makes “(a = a)” false. Since I makes “(a = a)” false, then there is some $\alpha \in D$ and some $\beta \in D$ such that I designates α by “a” and designates β by “a” and $\langle \alpha, \beta \rangle \notin “=”$ [TC =]. However, since “a” designates α , by substitution this means $\langle \alpha, \alpha \rangle \notin “=”$. But, by the *Definition of Identity*, for every $\alpha \in D$, $\langle \alpha, \alpha \rangle \in “=”$. This is a contradiction. Thus, the interpretation I is impossible. Hence, there is no interpretation that makes “(a = a)” false. Hence, “(a = a)” is L-true.

3. Show that “ $\forall x \exists y (x = y)$ ” is L-true.

Proof: Suppose not, so some I makes it false. So,

1. Some $\alpha \in D$ does not satisfy “ $\exists y (x = y)$ ” on I. [TC $\forall$]
 2. From 1., for some $\alpha \in D$, if ‘ $\mathfrak{m}$ ’ designates α on I, then ‘ $\exists y (\mathfrak{m} = y)$ ’ is false on I. [Def. Sat.]
 3. From 2., for some $\alpha \in D$, if ‘ $\mathfrak{m}$ ’ designates α on I, every $\beta \in D$ does not satisfy ‘ $(\mathfrak{m} = y)$ ’ on I. [TC $\exists$]
 4. From 3., for some $\alpha \in D$, every $\beta \in D$ is such that, if ‘ $\mathfrak{m}$ ’ designates α on I and ‘ $\mathfrak{n}$ ’ designates β on I, then ‘ $(\mathfrak{m} = \mathfrak{n})$ ’ is false on I. [Def. Sat.]
 5. So, for some $\alpha \in D$, every $\beta \in D$ is such that $\langle \alpha, \beta \rangle \notin “=”$ on I. [TC = / TC Atomic]
- Since β can be any object from D (β is unrestricted), let β be the object α . Then, by substitution (into 5.), $\langle \alpha, \alpha \rangle \notin “=”$. But for every $\alpha \in D$, $\langle \alpha, \alpha \rangle \in “=”$ [Def. Id.]. This is a contradiction. So, I is impossible. Hence, no interpretation makes it false. Hence, it is L-true.

4. Show that “ $\exists x \forall y (x \neq y)$ ” is L-false.

Proof: Suppose not, so some I makes it true. Then,

1. Some $\alpha \in D$ satisfies “ $\forall y (x \neq y)$ ” on I. [TC $\exists$]
2. From 1., for some $\alpha \in D$, if I ‘ $\mathfrak{m}$ ’ designates α on I, then ‘ $\forall y (\mathfrak{m} \neq y)$ ’ is true on I. [Def. Sat.]
3. From 2., for some $\alpha \in D$, if I ‘ $\mathfrak{m}$ ’ designates α on I, every $\beta \in D$ satisfies ‘ $(\mathfrak{m} \neq y)$ ’ on I. [TC $\forall$]
4. From 3., for some $\alpha \in D$, every $\beta \in D$ is such that, if I ‘ $\mathfrak{m}$ ’ designates α on I and ‘ $\mathfrak{n}$ ’ designates β on I, then ‘ $(\mathfrak{m} \neq \mathfrak{n})$ ’ is true on I. [Def. Sat.]
5. From 4., for some $\alpha \in D$, every $\beta \in D$ is such that, if I ‘ $\mathfrak{m}$ ’ designates α on I and ‘ $\mathfrak{n}$ ’ designates β on I, then ‘ $(\mathfrak{m} = \mathfrak{n})$ ’ is false on I. [TC $\neg$]
6. From 5., for some $\alpha \in D$, every $\beta \in D$ is such that $\langle \alpha, \beta \rangle \notin “=”$ on I. [TC =]

Since β is unrestricted (can be any object from D), let β be the object α . Then, by substitution into 6., $\langle \alpha, \alpha \rangle \notin “=”$. But for every $\alpha \in D$, $\langle \alpha, \alpha \rangle \in “=”$ on I. [Def. Id.] Contradiction. So, I is impossible. Hence, no interpretation makes it true. So, the sentence is L-false.

5. Show that “ $\forall x (F(x) \leftrightarrow \exists y (F(y) \wedge (x = y)))$ ” is L-true.

Proof: Suppose that it is not L-true. Then some I makes it false. Thus,

1. Some $\alpha \in D$ does not satisfy “ $(F(x) \leftrightarrow \exists y (F(y) \wedge (x = y)))$ ” on I. [TC $\forall$]
2. From 1., for some $\alpha \in D$, if ‘ $\mathfrak{m}$ ’ designates α on I, then ‘ $(F(\mathfrak{m}) \leftrightarrow \exists y (F(y) \wedge (\mathfrak{m} = y)))$ ’ is false on I. [Def. Sat.]

3. From 2., for some $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then either
 - i) I makes ' $F(\mathfrak{m})$ ' true AND makes ' $\exists y(F(y) \wedge (\mathfrak{m} = y))$ ' false OR
 - ii) I makes ' $F(\mathfrak{m})$ ' false AND makes ' $\exists y(F(y) \wedge (\mathfrak{m} = y))$ ' true. [TC $\leftrightarrow$]
4. From 3i), for some $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then $\alpha \in \text{"F"}$ on I [TC Atomic] AND every $\beta \in D$ does not satisfy ' $(F(y) \wedge (\mathfrak{m} = y))$ ' on I. [TC $\exists$]
5. From 4., for some $\alpha \in D$, every $\beta \in D$ is such that, if ' $\mathfrak{m}$ ' designates α on I and ' $\mathfrak{n}$ ' designates β on I, then $\alpha \in \text{"F"}$ on I AND ' $(F(\mathfrak{n}) \wedge (\mathfrak{m} = \mathfrak{n}))$ ' is false on I. [Def. Sat.]
6. From 5., for some $\alpha \in D$, every $\beta \in D$ is such that, if ' $\mathfrak{m}$ ' designates α on I and ' $\mathfrak{n}$ ' designates β on I, then $\alpha \in \text{"F"}$ on I AND either ' $F(\mathfrak{n})$ ' is false on I OR ' $(\mathfrak{m} = \mathfrak{n})$ ' is false on I. [TC $\wedge$]
7. From 6., for some $\alpha \in D$, every $\beta \in D$ is such that $\alpha \in \text{"F"}$ on I AND either $\beta \notin \text{"F"}$ on I [TC Atomic] OR $\langle \alpha, \beta \rangle \notin \text{"="}$ on I. [TC =]
 Since β is unrestricted, let β be the object α . Then, [by substitution] $\alpha \notin \text{"F"}$ on I OR $\langle \alpha, \alpha \rangle \notin \text{"="}$ on I. The first option contradicts the first part of 6., and the second condition contradicts Def. Id. So, either way, 3i) leads to contradiction.
8. From 3ii), for some $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then $\alpha \notin \text{"F"}$ on I [TC Atomic] AND some $\beta \in D$ satisfies ' $(F(y) \wedge (\mathfrak{m} = y))$ ' on I. [TC $\exists$]
9. From 8., for some $\alpha \in D$, some $\beta \in D$ is such that, if ' $\mathfrak{m}$ ' designates α on I and ' $\mathfrak{n}$ ' designates β on I, then then $\alpha \notin \text{"F"}$ on I AND ' $(F(\mathfrak{n}) \wedge (\mathfrak{m} = \mathfrak{n}))$ ' is true on I. [Def. Sat.]
10. From 9., for some $\alpha \in D$, some $\beta \in D$ is such that, if ' $\mathfrak{m}$ ' designates α on I and ' $\mathfrak{n}$ ' designates β on I, then $\alpha \notin \text{"F"}$ on I AND ' $F(\mathfrak{n})$ ' is true on I AND ' $(\mathfrak{m} = \mathfrak{n})$ ' is true on I. [TC $\wedge$]
11. From 10., for some $\alpha \in D$, some $\beta \in D$ is such that $\alpha \notin \text{"F"}$ on I AND $\beta \in \text{"F"}$ on I [TC Atomic] AND $\langle \alpha, \beta \rangle \in \text{"="}$ on I [TC =].

By substitution of α for β (since $\langle \alpha, \beta \rangle \in \text{"="}$) into the second part of 11., we get $\alpha \in \text{"F"}$ on I. But this contradicts the first part of 11. ($\alpha \notin \text{"F"}$ on I). So, 3ii) leads to contradiction. Since all cases lead to contradiction, I is impossible. So, no I makes it false. So, it is L-true.