

Soundness and Completeness
PHIL 422
Woodbridge

Let $\Sigma = \Gamma \cup \{\neg\phi\}$

Part I. Showing that Σ is consistent iff $\Gamma \not\models \phi$.

Suppose Σ is consistent.

$\Rightarrow \Gamma \cup \{\neg\phi\}$ is consistent

$\Rightarrow$ some I makes all of $\Gamma \cup \{\neg\phi\}$ true

$\Rightarrow$ some I makes all of Γ true and ' $\neg\phi$ ' true

$\Rightarrow$ some I makes all of Γ true and ' ϕ ' false [which is the definition of "entailment"]

$\Rightarrow \Gamma \not\models \phi$

$\therefore$ If Σ is consistent, then $\Gamma \not\models \phi$

Suppose Σ is inconsistent.

$\Rightarrow \Gamma \cup \{\neg\phi\}$ is inconsistent

$\Rightarrow$ no I makes all of $\Gamma \cup \{\neg\phi\}$ true

$\Rightarrow$ no I makes all of Γ true and ' $\neg\phi$ ' true

$\Rightarrow$ no I makes all of Γ true and ' ϕ ' false [TC $\neg$] [which is the definition of entailment]

$\Rightarrow \Gamma \models \phi$

$\therefore$ If Σ is inconsistent, then $\Gamma \models \phi$ [or what is the same: If $\Gamma \not\models \phi$, then Σ is consistent.]

Result I: Σ is consistent iff $\Gamma \not\models \phi$ [which is the same as: Σ is inconsistent iff $\Gamma \models \phi$.]

Part II. Let " $T[\Sigma]$ " stand for a completed tree for Σ . Show that $T[\Sigma]$ is open iff $\Gamma \not\models \phi$.

Let $\Sigma = \Gamma \cup \{\neg\phi\}$ and suppose $T[\Sigma]$ is open.

$\Rightarrow T[\Gamma \cup \{\neg\phi\}]$ is open [which is the definition of non-establishment]

$\Rightarrow \Gamma \not\models \phi$

$\therefore$ If $T[\Sigma]$ is open, then $\Gamma \not\models \phi$

Let $\Sigma = \Gamma \cup \{\neg\phi\}$ and suppose $T[\Sigma]$ is closed.

$\Rightarrow T[\Gamma \cup \{\neg\phi\}]$ is closed [which is the definition of establishment]

$\Rightarrow \Gamma \models \phi$

$\therefore$ If $T[\Sigma]$ is closed, then $\Gamma \models \phi$ [or what is the same as: If $\Gamma \not\models \phi$, then, $T[\Sigma]$ is open.]

Result II: $T[\Sigma]$ is open iff $\Gamma \not\models \phi$ [which is the same as: $T[\Sigma]$ is closed iff $\Gamma \models \phi$]

Part III. Now we will show that the Downward Adequacy Theorem (DAT) and the Upward Adequacy Theorem (UAT) holding for a formal system is equivalent to the system being both Sound and Complete (which we will label as “S/C”).

A formal system being both Sound (if $\Gamma \vdash \phi$, then $\Gamma \models \phi$) and Complete (if $\Gamma \models \phi$, then $\Gamma \vdash \phi$) together means the following:

$$\boxed{\text{S/C}} \quad \Gamma \models \phi \text{ iff } \Gamma \vdash \phi.$$

We will show this through an intermediate (logically equivalent) claim:

$$\boxed{\text{S/C}^*} \quad \Gamma \not\models \phi \text{ iff } \Gamma \not\vdash \phi.$$

DAT holding for a tree system means that for any arbitrary set of sentences, Σ , if Σ is consistent, then $T[\Sigma]$ will contain at least one open branch. So, if Σ is consistent, then $T[\Sigma]$ is open.

UAT holding for a tree system means that for any set of sentences, Σ , if Σ is inconsistent, then the $T[\Sigma]$ will contain no open branches. So, if Σ is inconsistent, then $T[\Sigma]$ is closed. This is equivalent to the claim that if $T[\Sigma]$ is not closed (i.e., is open), then Σ is not inconsistent (i.e., is consistent). Combining these theorems (to D/UAT) we have

$$\begin{array}{ccc} \boxed{\text{D/UAT}} & \Sigma \text{ is consistent} & \text{iff} & T[\Sigma] \text{ is open.} \\ & \Downarrow [\text{by } \boxed{\text{Result I}}] & & \Downarrow [\text{by } \boxed{\text{Result II}}] \\ & \Gamma \not\models \phi & \text{iff} & \Gamma \not\vdash \phi \\ & & \Downarrow [\text{by equivalence}] & \\ & \Gamma \models \phi & \text{iff} & \Gamma \vdash \phi \end{array}$$

$$\text{Thus, } \boxed{\text{D/UAT}} \Rightarrow \boxed{\text{S/C}^*} \Rightarrow \boxed{\text{S/C}}$$

$$\text{So, } \boxed{\text{D/UAT}} \Rightarrow \boxed{\text{S/C}}$$

Put another way: If D/UAT, then S/C. But, because the rules we employed are biconditionals, we could start with S/C and get to D/UAT. If you don’t see this, look at the proof from D/UAT to S/C, everywhere where there is a down-arrow showing biconditional equivalence, you could replace it with an up-arrow—because the claims are equivalent.

$$\text{That is: } \boxed{\text{S/C}} \Rightarrow \boxed{\text{S/C}^*} \Rightarrow \boxed{\text{D/UAT}}$$

$$\text{So, } \boxed{\text{S/C}} \Rightarrow \boxed{\text{D/UAT}}$$

But that means:

$$\boxed{\text{D/UAT}} \text{ iff } \boxed{\text{S/C}}$$

All of which is another way of saying that we prove the Soundness and Completeness of our system if and only if we prove DAT and UAT for our system! This is our culminating goal.

Proofs of the Soundness and Completeness of our system (S/C) that rely on D/UAT would proceed as follows.

IV. Proving Completeness: If $\Gamma \models \phi$, then $\Gamma \vdash \phi$.

Let $\Sigma = \Gamma \cup \{\neg\phi\}$

Suppose $\Gamma \models \phi$

$\Rightarrow$ no I makes all of Γ true and ' ϕ ' false [by the definition of semantic entailment]

$\Rightarrow$ no I makes all of Γ true and ' $\neg\phi$ ' true [TC $\neg$]

$\Rightarrow$ no I makes all of $\Gamma \cup \{\neg\phi\}$ true [from previous step and def. of " $\cup$ "]

$\Rightarrow$ no I makes all of Σ true [by $\Sigma = \Gamma \cup \{\neg\phi\}$]

$\Rightarrow \Sigma$ is inconsistent [by the definition of inconsistency]

$\Rightarrow T[\Sigma]$ is closed [by D/UAT]

$\Rightarrow T[\Gamma \cup \{\neg\phi\}]$ is closed [by substitution, since, $\Sigma = \Gamma \cup \{\neg\phi\}$]

$\Rightarrow \Gamma \vdash \phi$ [by the definition of establishment]

$\therefore$ If $\Gamma \models \phi$, then $\Gamma \vdash \phi$.

V. Proving Soundness: If $\Gamma \vdash \phi$, then $\Gamma \models \phi$.

Let $\Sigma = \Gamma \cup \{\neg\phi\}$

Suppose $\Gamma \vdash \phi$

$\Rightarrow T[\Gamma \cup \{\neg\phi\}]$ is closed [by the definition of establishment]

$\Rightarrow T[\Sigma]$ is closed [by the definition of Σ , etc.]

$\Rightarrow \Sigma$ is inconsistent [by D/UAT]

$\Rightarrow$ no I makes all of Σ true [by the definition of inconsistency]

$\Rightarrow$ no I makes all of $\Gamma \cup \{\neg\phi\}$ true [$\Sigma = \Gamma \cup \{\neg\phi\}$]

$\Rightarrow$ no I makes all of Γ true and ' $\neg\phi$ ' true [from previous step and def. of " $\cup$ "]

$\Rightarrow$ no I makes all of Γ true and ' ϕ ' false [TC $\neg$]

$\Rightarrow \Gamma \models \phi$ [by the definition of entailment].

$\therefore$ If $\Gamma \vdash \phi$, then $\Gamma \models \phi$.

VI. Now we can easily prove S/C by simply forming the biconditional from the results of proving Soundness and proving Completeness. The missing pieces are the proofs of DAT and UAT. Those proofs (by mathematical induction) are what we will work on for the rest of the course.