

PHIL 422: Sample Problems and Assignment #7 (due Sunday 10/18 by 10pm)

Sample Problems

1. Show " $\forall x(F(x) \rightarrow G(x))$ " is semantically equivalent to " $\forall y(\neg G(y) \rightarrow \neg F(y))$ "

Proof: Suppose not, then some I gives the sentences different truth-values. There are two possible cases.

Case 1: Some I makes

1. " $\forall x(F(x) \rightarrow G(x))$ " true AND
2. " $\forall y(\neg G(y) \rightarrow \neg F(y))$ " false.
3. From 1., **every** $\alpha \in D$ satisfies " $(F(x) \rightarrow G(x))$ " on I. [TC $\forall$]
4. From 3., for every $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then ' $(F(\mathfrak{m}) \rightarrow G(\mathfrak{m}))$ ' is true on I. [Def Sat]
5. From 4., either I makes ' $F(\mathfrak{m})$ ' false OR I makes ' $G(\mathfrak{m})$ ' true. [TC $\rightarrow$]
6. From 5., **for every** $\alpha \in D$, either $\alpha \notin "F"$ on I OR $\alpha \in "G"$ on I. [TC Atomic]
7. From 2., **some** $\gamma \in D$ does not satisfy " $(\neg G(y) \rightarrow \neg F(y))$ " on I. [TC $\forall$]
8. From 7., for some $\gamma \in D$, if ' $\mathfrak{n}$ ' designates γ on I, then ' $(\neg G(\mathfrak{n}) \rightarrow \neg F(\mathfrak{n}))$ ' is false on I. [Def Sat]
9. From 8., I makes ' $\neg G(\mathfrak{n})$ ' true AND makes ' $\neg F(\mathfrak{n})$ ' false. [TC $\rightarrow$]
10. From 9., I makes ' $G(\mathfrak{n})$ ' false AND makes ' $F(\mathfrak{n})$ ' true. [TC $\neg$]
11. From 10., **for some** $\gamma \in D$, $\gamma \notin "G"$ on I AND $\gamma \in "F"$ on I. [TC Atomic]
Since α is unrestricted, consider when α is γ . But then (by Substitution into 6.), either $\gamma \notin "F"$ on I OR $\gamma \in "G"$ on I. The first option contradicts $\gamma \in "F"$ on I (from 11.) and the second option contradicts $\gamma \notin "G"$ on I (from 11.) So, this case fails.

Case 2. Some I makes

1. " $\forall x(F(x) \rightarrow G(x))$ " false AND
2. " $\forall y(\neg G(y) \rightarrow \neg F(y))$ " true.
3. From 1., **some** $\alpha \in D$ does not satisfy " $(F(x) \rightarrow G(x))$ " on I. [TC $\forall$]
4. From 3., **for some** $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then ' $(F(\mathfrak{m}) \rightarrow G(\mathfrak{m}))$ ' is false on I. [Def Sat]
5. From 4., I makes ' $F(\mathfrak{m})$ ' true AND I makes ' $G(\mathfrak{m})$ ' false. [TC $\rightarrow$]
6. From 5., **for some** $\alpha \in D$, $\alpha \in "F"$ on I AND $\alpha \notin "G"$ on I. [TC Atomic]
7. From 2., **every** $\gamma \in D$ satisfies " $(\neg G(y) \rightarrow \neg F(y))$ " on I. [TC $\forall$]
8. From 7., **for every** $\gamma \in D$, if ' $\mathfrak{n}$ ' designates γ on I, then ' $(\neg G(\mathfrak{n}) \rightarrow \neg F(\mathfrak{n}))$ ' is true on I. [Def Sat]
9. From 8., either I makes ' $\neg G(\mathfrak{n})$ ' false OR I makes ' $\neg F(\mathfrak{n})$ ' true. [TC $\rightarrow$]
10. From 9., either I makes ' $G(\mathfrak{n})$ ' true OR I makes ' $F(\mathfrak{n})$ ' false. [TC $\neg$]
11. From 10., **for every** $\gamma \in D$, $\gamma \in "G"$ on I OR $\gamma \notin "F"$ on I. [TC Atomic]
Since γ is unrestricted, consider when γ is α . But then (by Substitution into 11.) we have $\alpha \in "G"$ on I OR $\alpha \notin "F"$ on I. The first contradicts $\alpha \notin "G"$ on I (from 6.) and the second contradicts $\alpha \in "F"$ on I (from 6.) So this case fails.

Since both cases fail, there are no interpretations that give these sentences different truth-values. So, these sentences are semantically equivalent.

2. Show $\{ \forall x \exists y R(x,y), \exists y \forall x R(x,y) \}$ is consistent.

Proof: If the set is consistent, then some I makes:

1. " $\forall x \exists y R(x,y)$ " true AND
2. " $\exists y \forall x R(x,y)$ " true.
3. From 1., **every** $\alpha \in D$ satisfies " $\exists y R(x,y)$ " on I. [TC $\forall$]
4. From 3., **for every** $\alpha \in D$, if ' m ' designates α on I, then ' $\exists y R(m,y)$ ' is true on I. [Def Sat.]
5. From 4., for every $\alpha \in D$, **some** $\beta \in D$ satisfies ' $R(m,y)$ ' on I. [TC $\exists$]
6. From 5., **for every** $\alpha \in D$, **there some** $\beta \in D$ such that, if ' n ' designates β on I, then ' $R(m,n)$ ' is true on I. [Def. Sat.]
7. From 6., for every $\alpha \in D$, there is some $\beta \in D$, such that $\langle \alpha, \beta \rangle \in "R"$ on I. [TC Atomic]
8. From 2., **some** $\gamma \in D$ satisfies " $\forall x R(x,y)$ " on I. [TC $\exists$]
9. From 8., **for some** $\gamma \in D$, if I designates γ by ' m ', then ' $\forall x R(x,m)$ ' is true on I. [Def. Sat.]
10. From 9., **for some** $\gamma \in D$, **every** $\alpha \in D$ satisfies ' $R(x,m)$ ' on I. [TC $\forall$]
11. From 10., **for some** $\gamma \in D$, **every** $\alpha \in D$ is such that, if ' n ' designates α on I, then ' $R(n,m)$ ' is true on I. [Def. Sat.]
12. From 11., for some $\gamma \in D$, every $\alpha \in D$ is such that $\langle \alpha, \gamma \rangle \in "R"$ on I. [TC Atomic]
By 7., every object in the domain, D, is such that there is some object in D, where an ordered pair of the former followed by the latter is in the extension of "R". By 11., there is some object in D such that for every object in D, a pair of the latter first, then the former is in the extension of "R". Since our recipe has two restricted introductions of objects, have I make $D = \{1,2\}$. Then we can fulfill both parts of the recipe if we have I assign "R" the extension $\{\langle 1,1 \rangle, \langle 2,1 \rangle\}$. [Note: it turns out that we could also have let I make $D=\{1\}$ and "R": $\{\langle 1,1 \rangle\}$, even though there are two restricted introductions of objects.] This I makes both sentences true, so the set is consistent.

3. Show " $\forall x \exists y \exists z F(x,y,z)$ " is contingent.

Proof. We need two interpretations, one, call it I_t , that makes the sentence true, the other, call it I_f , that makes it false.

Let I_t make

1. " $\forall x \exists y \exists z F(x,y,z)$ " true.
2. From 1., **every** $\alpha \in D$ satisfies " $\exists y \exists z F(x,y,z)$ " on I_t . [TC $\forall$]
3. From 2., **for every** $\alpha \in D$, if ' m ' designates α on I_t , then ' $\exists y \exists z F(m,y,z)$ ' is true on I_t . [Def Sat]
4. From 3., for every $\alpha \in D$, **some** $\beta \in D$ satisfies ' $\exists z F(m,y,z)$ ' on I_t . [TC $\exists$]
5. From 4., for every $\alpha \in D$, **there is some** $\beta \in D$ such that, if ' n ' designates β on I_t , then ' $\exists z F(m,n,z)$ ' is true on I_t . [Def Sat]
6. From 5., for every $\alpha \in D$, there is some $\beta \in D$, such that **some** $\gamma \in D$ satisfies ' $F(m,n,z)$ ' on I_t . [TC $\exists$]

7. From 6., for every $\alpha \in D$, there is some $\beta \in D$, such that **for some $\gamma \in D$** , if ' $\mathbf{k}$ ' designates γ on I_t , then ' $F(\mathbf{m}, \mathbf{n}, \mathbf{k})$ ' is true on I_t . [Def Sat]
8. From 7., for every $\alpha \in D$, there is a $\beta \in D$, such that there is some $\gamma \in D$ where $\langle \alpha, \beta, \gamma \rangle \in "F"$ on I_t . [TC Atomic].
 β and γ are particular individuals in D on I_t , while α is any individual in D . Since there are two restricted introductions of objects, have I_t set $D = \{1, 2\}$. Then have I_t assign " F " the extension: $\{\langle 1, 1, 2 \rangle, \langle 2, 1, 2 \rangle\}$. This makes the sentence true.

Let I_f make

1. " $\forall x \exists y \exists z F(x, y, z)$ " false.
2. From 1., **some $\alpha \in D$** does not satisfy " $\exists y \exists z F(x, y, z)$ " on I_f . [TC $\forall$]
3. From 2., **for some $\alpha \in D$** , if ' $\mathbf{m}$ ' designates α on I_f , then ' $\exists y \exists z F(\mathbf{m}, y, z)$ ' is false on I_f . [Def Sat]
4. From 3., for some $\alpha \in D$, **every $\beta \in D$** does not satisfy ' $\exists z F(\mathbf{m}, y, z)$ ' on I_f . [TC $\exists$]
5. From 4., some $\alpha \in D$ is such that, **for every $\beta \in D$** , if ' $\mathbf{n}$ ' designates β on I_f , then ' $\exists z F(\mathbf{m}, \mathbf{n}, z)$ ' is false on I_f . [Def Sat]
6. From 5., some $\alpha \in D$ is such that, every $\beta \in D$ is such that, **every $\gamma \in D$** does not satisfy ' $F(\mathbf{m}, \mathbf{n}, z)$ ' on I_f . [TC $\exists$]
7. From 6., some $\alpha \in D$ is such that, every $\beta \in D$ is such that, **for every $\gamma \in D$** , if ' $\mathbf{k}$ ' designates γ on I_f , then ' $F(\mathbf{m}, \mathbf{n}, \mathbf{k})$ ' is false on I_f . [Def Sat]
8. From 7., some $\alpha \in D$ is such that, every $\beta \in D$ is such that, for every $\gamma \in D$, $\langle \alpha, \beta, \gamma \rangle \notin "F"$ on I_f . [TC Atomic]
 Note that α is a particular member of D on I_f , while β and γ are any members of D on I_f (including α). So, consider when all the domain contains is α . Then we have the following I_f : $D = \{1\}$, " F ": $\{\}$. This makes the sentence false.

Thus, there is (at least) one interpretation, I_t , that makes the sentence true, and (at least) one interpretation, I_f , that makes it false. Hence, the sentence is contingent.

4. Show " $(\forall x \exists y \exists z F(x, y, z) \vee \neg \exists x \forall y \forall z F(x, y, z))$ " is contingent.

Proof. We need two interpretations, one, I_t , that makes the sentence true, the other, I_f , that makes it false.

Let I_t make

1. " $(\forall x \exists y \exists z F(x, y, z) \vee \neg \exists x \forall y \forall z F(x, y, z))$ " true.
2. From 1., either I makes " $\forall x \exists y \exists z F(x, y, z)$ " true OR
3. I makes " $\neg \exists x \forall y \forall z F(x, y, z)$ " true. [TC $\vee$]
 [Notice that we only need to make one side true to make the whole sentence true.]
4. From 2., **every $\alpha \in D$** satisfies " $\exists y \exists z F(x, y, z)$ " on I_t . [TC $\forall$]
5. From 4., **for every $\alpha \in D$** , if ' $\mathbf{m}$ ' designates α on I_t , then ' $\exists y \exists z F(\mathbf{m}, y, z)$ ' is true on I_t . [Def Sat]
6. From 5., for every $\alpha \in D$, **some $\beta \in D$** satisfies ' $\exists z F(\mathbf{m}, y, z)$ ' on I_t . [TC $\exists$]
7. From 6., for every $\alpha \in D$, **there is some $\beta \in D$** such that, if ' $\mathbf{n}$ ' designates β on I_t , then ' $\exists z F(\mathbf{m}, \mathbf{n}, z)$ ' is true on I_t . [Def Sat]
8. From 7., for every $\alpha \in D$, there is some $\beta \in D$ such that **some $\gamma \in D$** satisfies ' $F(\mathbf{m}, \mathbf{n}, z)$ ' on I_t . [TC $\exists$]

9. From 8., for every $\alpha \in D$, there is some $\beta \in D$ such that **there is some $\gamma \in D$** such that, if ' $\mathbf{k}$ ' designates γ on I_t , then ' $F(\mathbf{m}, \mathbf{n}, \mathbf{k})$ ' is true on I_t . [Def Sat]
10. From 9., for every $\alpha \in D$, there is some $\beta \in D$ such that there is some $\gamma \in D$ such that $\langle \alpha, \beta, \gamma \rangle \in "F"$ on I_t . [TC Atomic]
If we make I_t : $D = \{1, 2, 3\}$, " F ": $\{\langle 1, 2, 3 \rangle, \langle 2, 2, 3 \rangle, \langle 3, 2, 3 \rangle\}$, this makes the first disjunct true, so it makes the whole sentence true. [Note: I made the domain contain three objects, for α , β , and γ , but an I with only two (one for each restricted β and γ) would work—so we could have I set $D = \{1, 2\}$ and " F ": $\{\langle 1, 2, 1 \rangle, \langle 2, 1, 2 \rangle\}$. In fact an I with only one object ($D = \{1\}$) would work here as well, with " F ": $\{\langle 1, 1, 1 \rangle\}$.]

Let I_f make " $(\forall x \exists y \exists z F(x, y, z) \vee \neg \exists x \forall y \forall z F(x, y, z))$ " false. Then I makes

1. " $\forall x \exists y \exists z F(x, y, z)$ " false AND
2. " $\neg \exists x \forall y \forall z F(x, y, z)$ " false. [TC $\vee$]
3. From 2., I_f makes " $\exists x \forall y \forall z F(x, y, z)$ " true. [TC $\neg$]
4. From 1., **some $\alpha \in D$** does not satisfy " $\exists y \exists z F(x, y, z)$ " on I_f . [TC $\forall$]
5. From 4., **for some $\alpha \in D$** , if ' $\mathbf{m}$ ' designates α on I_f , then ' $\exists y \exists z F(\mathbf{m}, y, z)$ ' is false on I_f . [Def Sat]
6. From 5., for some $\alpha \in D$, **every $\beta \in D$** does not satisfy ' $\exists z F(\mathbf{m}, y, z)$ ' on I_f . [TC $\exists$]
7. From 6., for some $\alpha \in D$, **for every $\beta \in D$** , if ' $\mathbf{n}$ ' designates β on I_f , then ' $\exists z F(\mathbf{m}, \mathbf{n}, z)$ ' is false on I_f . [Def Sat]
8. From 7., for some $\alpha \in D$, every $\beta \in D$ is such that **every $\gamma \in D$** does not satisfy ' $F(\mathbf{m}, \mathbf{n}, z)$ ' on I_f . [TC $\exists$]
9. From 8., for some $\alpha \in D$, every $\beta \in D$ is such that, **for every $\gamma \in D$** , if ' $\mathbf{k}$ ' designates γ on I_f , then ' $F(\mathbf{m}, \mathbf{n}, \mathbf{k})$ ' is false on I_f . [Def Sat]
10. From 9., for some $\alpha \in D$, every $\beta \in D$ is such that, for every $\gamma \in D$, $\langle \alpha, \beta, \gamma \rangle \notin "F"$ on I_f . [TC Atomic]
11. From 3., **some $\delta \in D$** satisfies " $\forall y \forall z F(x, y, z)$ " on I_f . [TC $\exists$]
12. From 11., **for some $\delta \in D$** , if ' $\mathbf{m}$ ' designates δ on I_f , then ' $\forall y \forall z F(\mathbf{m}, y, z)$ ' is true on I_f . [Def Sat]
13. From 12., for some $\delta \in D$, **every $\rho \in D$** satisfies ' $\forall z F(\mathbf{m}, y, z)$ ' on I_f . [TC $\forall$]
14. From 13., some $\delta \in D$ is such that, **for every $\rho \in D$** , if ' $\mathbf{n}$ ' designates ρ on I_f , then ' $\forall z F(\mathbf{m}, \mathbf{n}, z)$ ' is true on I_f . [Def Sat]
15. From 14., some $\delta \in D$ is such that, for every $\rho \in D$, **every $\sigma \in D$** satisfies ' $F(\mathbf{m}, \mathbf{n}, z)$ ' on I_f . [TC $\forall$]
16. From 15., some $\delta \in D$ is such that, for every $\rho \in D$, **every $\sigma \in D$** is such that, if ' $\mathbf{k}$ ' designates ρ on I_f , then ' $F(\mathbf{m}, \mathbf{n}, \mathbf{k})$ ' is true on I_f . [Def Sat]
17. From 16., some $\delta \in D$ is such that, for every $\rho \in D$, every $\sigma \in D$ is such that, $\langle \delta, \rho, \sigma \rangle \in "F"$ on I_f . [TC Atomic]

We will build our interpretation out of 10. and 16. Here is I_f . Notice that in fact 10. and 16., there are two restricted individuals in D on I_f , namely α and δ . All of the other objects are unrestricted in the domain. Hence, we can get by with just two individuals. Make I_f : $D = \{1, 2\}$ and " F ": $\{\langle 2, 1, 1 \rangle, \langle 2, 1, 2 \rangle, \langle 2, 2, 1 \rangle, \langle 2, 2, 2 \rangle\}$. This interpretation makes the sentence false. [Note: I had left the red triples out, but they need to be in the extension so one δ leads a triple for every ρ combined with every σ .]

Thus, there is (at least) one interpretation, I_t , that makes the sentence true, and (at least) one interpretation, I_f , that makes it false. Hence, the sentence is contingent.

Assigned Problems

5. Show $\{ \forall x F(x, b) \} \models \forall x \exists y F(x, y)$.
6. Show “ $(\forall x (F(x) \rightarrow G(x)) \rightarrow \exists x (F(x) \wedge G(x)))$ ” is contingent.
7. Show $\{ \forall x (F(x) \rightarrow G(x, x)), \forall x \forall y (G(x, x) \rightarrow \neg F(y)) \}$ is consistent.
8. Show $\{ \forall x \exists y \exists z R(x, y, z) \} \not\models \exists y \exists z \forall x R(x, y, z)$