

**PHIL 422: Examples, Practice Problems, and
Assignment #6 (I.2, II.3, III.2, IV.2) – due Tuesday 10/13 by 10pm**

I. For each of the following sentences, show that it has the semantic status claimed.

1. Prove “ $(\exists x(F(x) \wedge G(x)) \wedge (\neg \exists y F(y) \wedge \exists z G(z)))$ ” is L-false.
Proof: Suppose not; i.e., suppose some I makes it true. So,
 1. I makes “ $\exists x(F(x) \wedge G(x))$ ” true AND
 2. I makes “ $(\neg \exists y F(y) \wedge \exists z G(z))$ ” true. [TC $\wedge$]
 3. From 1., **for some** $\alpha \in D$, α satisfies “ $(F(x) \wedge G(x))$ ” on I. [TC $\exists$]
 4. From 3., if ‘**m**’ designates α on I, then I makes ‘ $(F(\mathbf{m}) \wedge G(\mathbf{m}))$ ’ true. [Def. Sat.]
 5. From 4., I makes ‘ $F(\mathbf{m})$ ’ true AND makes ‘ $G(\mathbf{m})$ ’ true. [TC $\wedge$]
 6. From 5., $\alpha \in$ “F” on I AND $\alpha \in$ “G” on I. [TC Atomic]
 7. From 2., I makes “ $\neg \exists y F(y)$ ” true AND makes “ $\exists z G(z)$ ” true. [TC $\wedge$]
 8. From 7., I makes “ $\exists y F(y)$ ” false. [TC $\neg$]
 9. From 8., **for every** $\beta \in D$, β does not satisfy “ $F(y)$ ” on I. [TC $\exists$]
 10. From 9., if ‘**n**’ designates β on I, then I makes ‘ $F(\mathbf{n})$ ’ false. [Def. Sat.]
 11. From 10., $\beta \notin$ “F” on I. [TC Atomic]
 Since β is **unrestricted** (can be **any object** from D), and α is **restricted** (to just “some” object from D), consider when β is α . Then (by 11.) $\alpha \notin$ “F” on I. [Substitution] This contradicts 6. So, I is impossible. So, no I makes the sentence true. So, it is L-false.
2. Prove “ $\forall x(H(x,a) \rightarrow K(a,x)) \rightarrow (H(b,a) \rightarrow K(a,b))$ ” is L-true.
3. Prove “ $(\forall x F(x) \vee \exists y \neg F(y))$ ” is L-true.
4. Prove “ $(\forall x(F(x) \rightarrow G(x)) \wedge \exists y(F(y) \wedge \neg G(y)))$ ” is L-false.

II. Show that the following entailments hold.

1. $\{\forall x(F(x) \rightarrow G(x)), \forall y(G(y) \rightarrow H(y)), \forall z(\neg J(z) \rightarrow \neg H(z))\} \models (\exists x F(x) \rightarrow \exists w J(w))$
Proof: Suppose not, that is, suppose some I makes:
 1. “ $\forall x(F(x) \rightarrow G(x))$ ” true
 2. “ $\forall y(G(y) \rightarrow H(y))$ ” true
 3. “ $\forall z(\neg J(z) \rightarrow \neg H(z))$ ” true
 4. “ $(\exists x F(x) \rightarrow \exists w J(w))$ ” false
 5. From 1., **for every** $\alpha \in D$, α satisfies “ $(F(x) \rightarrow G(x))$ ” on I. [TC $\forall$]
 6. From 5., I makes ‘ $(F(\mathbf{m}) \rightarrow G(\mathbf{m}))$ ’ true, if ‘**m**’ designates α on I. [Def. Sat.]
 7. From 6., I makes ‘ $F(\mathbf{m})$ ’ false OR ‘ $G(\mathbf{m})$ ’ true. [TC $\rightarrow$]
 8. From 7., $\alpha \notin$ “F” on I OR $\alpha \in$ “G” on I. [TC Atomic]
 9. From 2., **for every** $\beta \in D$, β satisfies “ $(G(y) \rightarrow H(y))$ ” on I. [TC $\forall$]
 10. From 9., I makes ‘ $(G(\mathbf{m}) \rightarrow H(\mathbf{m}))$ ’ true, if ‘**m**’ designates β on I. [Def. Sat.]
 11. From 10., I makes ‘ $G(\mathbf{m})$ ’ false OR makes ‘ $H(\mathbf{m})$ ’ true. [TC $\rightarrow$]
 12. From 11., $\beta \notin$ “G” on I OR $\beta \in$ “H” on I. [TC Atomic].
 13. From 3., **for every** $\gamma \in D$, γ satisfies “ $(\neg J(z) \rightarrow \neg H(z))$ ” on I. [TC $\forall$]
 14. From 13., I makes ‘ $(\neg J(\mathbf{m}) \rightarrow \neg H(\mathbf{m}))$ ’ true, if ‘**m**’ designates γ on I. [Def. Sat.]

15. From 14., I makes ' $\neg J(m)$ ' false OR ' $\neg H(m)$ ' true. [TC $\rightarrow$]
16. From 15., I makes ' $J(m)$ ' true OR makes ' $H(m)$ ' false. [TC $\neg$]
17. From 16., $\gamma \in "J"$ on I OR $\gamma \notin "H"$ on I. [TC Atomic]
18. From 4., I makes " $\exists x F(x)$ " true AND makes " $\exists w J(w)$ " false. [TC $\rightarrow$]
19. From 18., **for some** $\delta \in D$, δ satisfies " $F(x)$ " on I. [TC $\exists$]
20. From 19., if I designates δ by ' m ', then I makes ' $F(m)$ ' true. [Def. Sat.]
21. From 20., $\delta \in "F"$ on I. [TC Atomic]

Since α , β , and γ are **unrestricted** (can be any object from D), consider when they are all δ (since it is restricted to "some"). So (by 8.) $\delta \notin "F"$ on I OR $\delta \in "G"$ on I, and (by 12.) $\delta \notin "G"$ on I OR $\delta \in "H"$ on I, and (by 17.) $\delta \in "J"$ on I OR $\delta \notin "H"$ on I. [Substitution]
By 21. and 8., $\delta \in "G"$ on I. But then (by 12.) $\delta \in "H"$ on I. So,

22. From 17., $\delta \in "J"$ on I. [Substitution]
 23. From 18., **for every** $\sigma \in D$, σ does not satisfy " $J(w)$ " on I. [TC $\exists$]
 24. From 23., I makes ' $J(m)$ ' false, if ' m ' designates σ on I. [Def. Sat.]
 25. From 24., $\sigma \notin "J"$ on I. [TC Atomic] Since σ is unrestricted, consider when σ is δ .
 26. From 25., $\delta \notin "J"$ on I. [Substitution]
- But 26. contradicts 22. So I is impossible. So, no I is a counterexample (makes all the lhs sentence true and the rhs sentence false). So, the entailment holds.

2. $\{(F(a) \wedge \forall x(F(x) \rightarrow G(x)))\} \models G(a)$
3. $\{(F(a) \vee F(b)), \forall x(F(x) \rightarrow G(x))\} \models (G(a) \vee G(b))$

III. Inconsistency

1. Show $\{ F(a), F(b), F(c), \forall x(F(x) \rightarrow ((x = a) \vee (x = b))), (c \neq a), (b \neq c) \}$ is inconsistent.
Proof: Suppose not; i.e., suppose there is an interpretation that makes all of the members of the set true. That is, I makes:

1. " $F(a)$ " true
2. " $F(b)$ " true
3. " $F(c)$ " true
4. " $\forall x(F(x) \rightarrow ((x = a) \vee (x = b)))$ " true
5. " $(c \neq a)$ " true
6. " $(b \neq c)$ " true $\Rightarrow$ " b " does not des. γ and " c " does not des. β , i.e., $\langle \gamma, \beta \rangle \notin "="$

Also, there is some $\alpha \in D$ where " a " designate α on I, and there is some $\beta \in D$ where " b " designates β on I, and there is some $\gamma \in D$ where " c " designate γ on I.

7. From 1., $\alpha \in "F"$ on I. [TC Atomic]
8. From 2., $\beta \in "F"$ on I. [TC Atomic]
9. From 3., $\gamma \in "F"$ on I. [TC Atomic]
10. From 4., for every $\sigma \in D$, σ satisfies " $(F(x) \rightarrow ((x = a) \vee (x = b)))$ " on I. [TC $\forall$]
11. From 10., if ' u ' designates σ on I, then ' $(F(u) \rightarrow ((u = a) \vee (u = b)))$ ' is true on I. [Def Sat.]
12. From 11., I makes ' $F(u)$ ' false OR makes ' $((u = a) \vee (u = b))$ ' true. [TC $\rightarrow$]
13. From 12., $\sigma \notin "F"$ on I [TC Atomic] OR I makes ' $(u = a)$ ' true or ' $(u = b)$ ' true. [TC $\vee$]

14. From 13., $\sigma \notin \text{"F"}$ on I OR " a " designates σ on I or " b " designates σ on I. [TC =]

15. From 14., $\sigma \notin \text{"F"}$ on I OR $\langle \sigma, \alpha \rangle \in \text{"="}$ or $\langle \sigma, \beta \rangle \in \text{"="}$. [TC =]

Since σ is **unrestricted** in D, consider when σ is γ .

16. From 9., $\sigma \in \text{"F"}$ on I. [Substitution]

17. From 15. and 16., $\langle \sigma, \alpha \rangle \in \text{"="}$ or $\langle \sigma, \beta \rangle \in \text{"="}$.

18. From 5., " $(c = a)$ " is false on I. [TC $\neg$]

19. From 18., $\langle \gamma, \alpha \rangle \notin \text{"="}$. [TC =] So, $\langle \sigma, \alpha \rangle \notin \text{"="}$. [Substitution]

20. From 19. and 17., $\langle \sigma, \beta \rangle \in \text{"="}$.

21. From 6., " $(b = c)$ " is false on I. [TC $\neg$]

22. From 21., $\langle \beta, \gamma \rangle \notin \text{"="}$. [TC =] So, $\langle \beta, \sigma \rangle \notin \text{"="}$. [Substitution]

23. From 22., $\langle \sigma, \beta \rangle \notin \text{"="}$. [TC =]

But 20. and 23. contradict. So, I is impossible. So, no I can make all of the members of the set true at once. So, the set is inconsistent.

2. Show $\{ \forall x(F(x) \rightarrow G(x)), \exists y(F(y) \wedge H(y)), \neg \exists z G(z) \}$ is inconsistent.

IV. Equivalence

1. Prove " $\forall x(a = x)$ " is logically equivalent to " $\forall z(z = a)$ ".

Proof: Suppose not, i.e., suppose that, for some I, either I makes " $\forall x(a = x)$ " true and " $\forall z(z = a)$ " false OR I makes " $\forall x(a = x)$ " false and " $\forall z(z = a)$ " true. Also, for some $\alpha \in D$, " a " designates α on I.

Case 1: I makes 1. " $\forall x(a = x)$ " true and 2. " $\forall z(z = a)$ " false.

3. From 1., **for every** $\beta \in D$, β satisfies " $(a = x)$ " on I. [TC $\forall$]

4. If ' m ' designates β on I, then I makes ' $(a = m)$ ' true. [Def. Sat.]

5. From 2., **for some** $\gamma \in D$, γ does not satisfy " $(z = a)$ " on I. [TC $\forall$]

6. If I ' n ' designates γ on I, then I makes ' $(n = a)$ ' false. [Def. Sat.]

7. From 4., " a " designates β on I. [TC =]

8. From 6., " a " does not designate γ on I. [TC =]

9. From 7., $\langle \alpha, \beta \rangle \in \text{"="}$. [TC =]

10. From 9., $\langle \gamma, \alpha \rangle \notin \text{"="}$. [TC =]

Since β is **unrestricted** (can be **any object** in D), consider when β is γ (which is restricted).

11. From 9., $\langle \alpha, \gamma \rangle \in \text{"="}$. [Substitution]. But this means that α and γ are the same object.

[TC =] So, from 10., $\langle \gamma, \gamma \rangle \notin \text{"="}$. [Substitution] But that's impossible [Def. Id.].

So, this case fails.

Case 2: I makes 1. " $\forall x(a = x)$ " false and 2. " $\forall z(z = a)$ " true.

3. From 1., **for some** $\beta \in D$, β does not satisfy " $(a = x)$ " on I. [TC $\forall$]

4. From 3., if I designates β by ' m ', then I makes ' $(a = m)$ ' false. [Def. Sat.]

5. From 2., **for every** $\gamma \in D$, γ satisfies " $(z = a)$ " on I. [TC $\forall$]

6. From 5., if I designates γ by ' n ', then I makes ' $(n = a)$ ' true. [Def. Sat.]

7. From 4., " a " does not designates β on I. [TC =].

8. From 6., " a " does designate γ on I. [TC =]

9. From 7., $\langle \alpha, \beta \rangle \notin \text{"="}$. [TC =]

10. From 8., $\langle \gamma, \alpha \rangle \in "="$. [TC =]

Since γ is **unrestricted** (can be any object in D), consider when γ is β .

11. From 9., $\langle \alpha, \gamma \rangle \notin "="$. [Substitution]

But from 10., it follows that γ and α are the same object. [TC =] So, from 11., $\langle \gamma, \gamma \rangle \notin "="$. [Substitution] But that's impossible. [Def. "="] So, this case fails.

Since both cases fail, there is no I that makes these sentences have different truth values.

So, these sentences are logically equivalent.

2. Prove " $F(a) \wedge (b = a)$ " is logically equivalent to " $F(b) \wedge (a = b)$ ".