

Examples, Practice Problems, and

Assignment #5 (I.2, II.A.2, II.B.1, II.C.3, II.D.2 due Tues 10/6 by 10pm)

PHIL 422

Woodbridge

I. Showing Expressive Capacity: Demonstrating that FOL expresses/encodes some arbitrary truth-function by constructing a DNF sentence of FOL from the function.

1. **Example:** Consider the following truth-function, which amounts to a specific pattern of truth-value outputs for the complete set of ordered sequences of truth-values of size 3 as inputs.

<u>Input sequences</u>	<u>Truth-value outputs</u>
$\langle T, T, T \rangle$	$\Rightarrow$ F
$\langle T, T, F \rangle$	$\Rightarrow$ T
$\langle T, F, T \rangle$	$\Rightarrow$ T
$\langle T, F, F \rangle$	$\Rightarrow$ F
$\langle F, T, T \rangle$	$\Rightarrow$ T
$\langle F, T, F \rangle$	$\Rightarrow$ F
$\langle F, F, T \rangle$	$\Rightarrow$ T
$\langle F, F, F \rangle$	$\Rightarrow$ F

Think of the input sequences as forming the reference columns of a truth-table, with the atomic FOL sentence “A” over the first position, “B” over the second, and “C” over the third. Now look at the “row” where the output of the truth-function is T. For this truth-function it is “rows” 2, 3, 5, and 7. (You can ignore the rest as implicitly covered, since anywhere a truth-function does not kick out T, it will kick out F.) Look at the specific input sequences that yield the output T. If it has a T in the first position, write down “A”. If it has F in the first position, write down “ $\neg A$ ”. Follow this with a caret (“ $\wedge$ ”) and then write either “B” or “ $\neg B$ ”, depending on whether there is a T or F in the second position in the input sequence. Follow this with another “ $\wedge$ ” and then write down either “C” or “ $\neg C$ ”, depending on whether there is a T or F in the third position in the sequences. Put parentheses around the whole conjunction. Given the standard format of a truth-table, the rows in the reference columns can each be represented by a standardized conjunction in this way. Row 2 in a size-3 truth-table is always captured by “ $(A \wedge B \wedge \neg C)$ ”. For the above truth-table we want to capture all the input sequences with output T, so we also write down “ $(A \wedge \neg B \wedge C)$ ” for row 3, “ $(\neg A \wedge B \wedge C)$ ” for row 5, and “ $(\neg A \wedge \neg B \wedge C)$ ” for row 7. Then put them all together inside parentheses with wedges (“ $\vee$ ”) between them, to form the following.

$$((A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge C))$$

This sentence of FOL is in DNF and expresses the truth-function it is extracted from. Its standard form truth-table will have the same truth-value pattern in its final column as the truth-function.

2. **Problem:** Construct a DNF sentence of FOL (using the atomic sentence letters, “A”, “B”, “C”, “D”) that expresses the following size-4 truth-function.

$$\begin{aligned} \langle T, T, T, T \rangle &\Rightarrow T, \langle T, T, T, F \rangle \Rightarrow F, \langle T, T, F, T \rangle \Rightarrow F, \langle T, T, F, F \rangle \Rightarrow F, \\ \langle T, F, T, T \rangle &\Rightarrow F, \langle T, F, T, F \rangle \Rightarrow T, \langle T, F, F, T \rangle \Rightarrow F, \langle T, F, F, F \rangle \Rightarrow T, \\ \langle F, T, T, T \rangle &\Rightarrow F, \langle F, T, T, F \rangle \Rightarrow T, \langle F, T, F, T \rangle \Rightarrow F, \langle F, T, F, F \rangle \Rightarrow F, \\ \langle F, F, T, T \rangle &\Rightarrow T, \langle F, F, T, F \rangle \Rightarrow F, \langle F, F, F, T \rangle \Rightarrow F, \langle F, F, F, F \rangle \Rightarrow T \end{aligned}$$

II. Proofs of Global Semantic Facts

Examples

1. “ $((F(a) \wedge (a = b)) \wedge (b = c)) \rightarrow F(c)$ ” is L-true.

Proof: Suppose not; i.e., suppose some I makes it false.

So, I makes “ $((F(a) \wedge (a = b)) \wedge (b = c))$ ” true [①] AND I makes “ $F(c)$ ” false [②][TC $\rightarrow$].

By ①: I makes “ $F(a)$ ” true [③] AND “ $(a = b)$ ” true [④] AND “ $(b = c)$ ” true [TC $\wedge$].[⑤]

By ②: on I there is some $\alpha \in D$ such that “ c ” designates α , and $\alpha \notin “F”$ [TC Atomic].[⑥]

By ⑤ and ⑥: “ b ” designates α on I [TC $=$].[⑦]

By ④ and ⑦: “ a ” designates α on I [TC $=$].[⑧]

By ③ and ⑧: $\alpha \in “F”$ on I [TC Atomic].[⑨]

But, by ② and ⑨, now we have $\alpha \in “F”$ AND $\alpha \notin “F”$. That’s a contradiction.

So, I is impossible.

So, no I makes “ $((F(a) \wedge (a = b)) \wedge (b = c)) \rightarrow F(c)$ ” false.

So, the sentence is L-true.

2. Prove that $\{F(a), G(a), (a = b)\} \models (F(b) \rightarrow G(b))$

Proof: Suppose not, that is, suppose that some I makes:

[①] “ $F(a)$ ” true

[②] “ $G(a)$ ” true

[③] “ $(a = b)$ ” true

[④] “ $(F(b) \rightarrow G(b))$ ” false.

By ③: on I there is some $\alpha \in D$ such that both “ a ” and “ b ” designate α .[⑤]

By ① and designation from ⑤: $\alpha \in “F”$ [TC Atomic]

By ② and designation from ⑤: $\alpha \in “G”$ [TC Atomic]

By ④: I makes “ $F(b)$ ” true AND “ $G(b)$ ” false [TC $\rightarrow$][⑥].

By ⑥ and designation from ③: $\alpha \in “F”$ AND $\alpha \notin “G”$. [TC Atomic]

But this means that $\alpha \in “G”$ AND $\alpha \notin “G”$. But that’s impossible.

So, no I is a counterexample.

So, the entailment holds.

3. Prove that “ $(R(a,b) \wedge R(b,a))$ ” is semantically equivalent to “ $(R(b,a) \wedge R(a,b))$ ”

Proof: Suppose not; i.e., suppose that some I either makes “ $(R(a,b) \wedge R(b,a))$ ” true and makes “ $(R(b,a) \wedge R(a,b))$ ” false OR makes “ $(R(a,b) \wedge R(b,a))$ ” false and “ $(R(b,a) \wedge R(a,b))$ ” true.

Case 1: I makes “ $(R(a,b) \wedge R(b,a))$ ” true and “ $(R(b,a) \wedge R(a,b))$ ” false.

$\Rightarrow$ I makes “ $R(a,b)$ ” true and “ $R(b,a)$ ” true AND either “ $R(b,a)$ ” or “ $R(a,b)$ ” false. [TC $\wedge$]

$\Rightarrow$ on I there is some $\alpha \in D$ and some $\beta \in D$ such that

“ a ” designates α AND

“ b ” designates β AND

$\langle \alpha, \beta \rangle \in “R”$ [TC Atomic] AND

$\langle \beta, \alpha \rangle \in “R”$ [TC Atomic] AND

either $\langle \beta, \alpha \rangle \notin “R”$ OR $\langle \alpha, \beta \rangle \notin “R”$ [TC Atomic]. But both of these contradict.

So, this case fails.

Case 2: I makes " $(R(a,b) \wedge R(b,a))$ " false and " $(R(b,a) \wedge R(a,b))$ " true.

$\Rightarrow$ I makes either " $R(a,b)$ " or " $R(b,a)$ " false AND " $R(b,a)$ " true and " $R(a,b)$ " true. [TC $\wedge$]

$\Rightarrow$ on I there is some $\alpha \in D$ and some $\beta \in D$ such that

"a" designates α AND

"b" designates β AND

either $\langle \alpha, \beta \rangle \notin "R"$ OR $\langle \beta, \alpha \rangle \notin "R"$ [TC Atomic] AND

$\langle \beta, \alpha \rangle \in "R"$ [TC Atomic] AND

$\langle \alpha, \beta \rangle \in "R"$ [TC Atomic]. But the latter two contradict both of prior options.

So, this case fails.

Problems

A. Logical Truth. Show that each of the following is logically true.

1. $(F(a,b) \rightarrow \neg(G(d,e,c) \wedge \neg F(a,b)))$
2. $(A(a) \vee \neg(B(c) \wedge \neg \neg A(a)))$
3. $(((((A(a) \wedge B(b)) \vee (A(a) \wedge \neg B(b))) \vee (\neg A(a) \wedge B(b))) \vee (\neg A(a) \wedge \neg B(b)))$
4. $((F(b) \wedge (a = b)) \rightarrow F(a))$

B. Logically False. Show that each of the following is logically false.

1. $\neg(F(a,b) \rightarrow (G(a,b) \vee F(a,b)))$
2. $((F(a) \wedge \neg F(b)) \wedge (a = b))$

C. Entailment. Show that the following entailments hold.

1. $\{(F(a) \wedge F(b)), \neg F(b)\} \models G(a)$
2. $\{(F(a,b) \vee G(c,d)), (F(a,b) \rightarrow A(a)), (G(c,d) \rightarrow B(b))\} \models (A(a) \vee B(b))$
3. $\{(C(a,b) \vee C(b,c)), (b = c), \neg C(a,c)\} \models C(b,c)$

D. Inconsistency. Show that the following are inconsistent.

1. $\{(A(a) \vee A(b)), (A(a) \rightarrow G(c)), (G(c) \rightarrow \neg A(b)), (A(b) \rightarrow G(c)), (G(c) \rightarrow \neg A(a))\}$
2. $\{(A(a) \vee A(b)), (A(a) \rightarrow C(d)), (A(b) \rightarrow C(d)), \neg(C(d) \vee C(e))\}$
3. $\{(a = b), (b = c), (A(a) \wedge \neg A(c))\}$