

Examples, Practice Problems, and Assignment #4
 PHIL 422
 Woodbridge

For Assignment #4, hand in your answers to I.2, I.6, II.2, II.3, II.4, III.4, III.5, IV.2, IV.4, V.2 on Sunday 9/20 by 10pm. Submit your answers as a PDF file with a file name as follows ‘yourlastname HW4.pdf’ uploaded to the Submission link in Canvas (i.e., do NOT email me a link to a Google Drive).

I. Basic Syntax: For each of the following, use the formation rules to demonstrate first whether it is a wff, then say whether it is a *sentence*, of FOL. Explain your answer.

1. $(F(a) \wedge (a = c)) \rightarrow \forall x F(x)$
2. $(\forall x(F(x) \rightarrow G(x)) \vee \exists y(F(y) \wedge \neg G(y)))$
3. $\forall x \forall y((F(x) \wedge F(y)) \rightarrow (x = y))$
4. $\forall z(G(y) \rightarrow H(y,z))$
5. $\forall x \forall y(R(x = z) \vee \neg R(x = z))$
6. $((a \neq c) \rightarrow (b = d)) \leftrightarrow \forall x \neg \exists z F(x,z)$ [“ $\neq$ ” is accepted as the negation of an identity claim]

II. Basic Semantics 1: Semantic Assessment via Truth-Tables.

1) **Example:** Using a truth-table determine whether $\{((A \vee \neg C) \vee B)\} \models (\neg(B \leftrightarrow C) \vee A)$. Here is the truth table for these sentences [final columns in red].

A	B	C	$((A \vee \neg C) \vee B)$			$(\neg(B \leftrightarrow C) \vee A)$		
T	T	T	T	F	T	F	T	T
T	T	F	T	T	T	T	F	T
T	F	T	T	F	T	T	F	T
T	F	F	T	T	T	F	T	T
F	T	T	F	F	F	F	T	F
F	T	F	T	T	T	T	F	T
F	F	T	F	F	F	T	F	T
F	F	F	T	T	T	F	T	F

Since the left-hand-side (lhs) sentence is true with the right-hand-side (rhs) sentence is false in row 5 (as well as in row 8, but just one row—i.e., interpretation of the atomic sentences—doing this is enough), there is entailment, not entailment.

Problems: Using **truth-tables**, determine whether the queried semantic status or relation holds.

- 2) Are $\{ (P \wedge \neg Q), (Q \rightarrow R), (\neg R \rightarrow \neg P) \}$ consistent?
- 3) Is $\neg((A \vee B) \leftrightarrow (B \vee C)) \vee ((B \vee C) \rightarrow (A \vee C))$ contingent?
- 4) Show that, for any ‘ ϕ ’ and ‘ ψ ’, ‘ $(\phi \vee \psi)$ ’ is logically equivalent to ‘ $\neg(\neg\phi \wedge \neg\psi)$ ’
- 5) Show that, for any ‘ ϕ ’ and ‘ ψ ’, there are forms using only the connectives “ $\neg$ ”, “ $\wedge$ ”, and “ $\vee$ ” that are logically equivalent to ‘ $(\phi \rightarrow \psi)$ ’ and to ‘ $(\phi \leftrightarrow \psi)$ ’.

III. **Basic Semantics 2:** Semantic Assessment via Sequent Analysis (i.e., where you try to construct a particular row from a truth table—an interpretation—that shows something specific).

1) **Example:** Here is an example, testing an entailment that involves four atomic sentence letter types, meaning the truth-table would have 16 rows (and note that a 5-letter case would involve a 32 row truth-table—something no one wants to construct!).

$$\begin{array}{ccccccc} & \text{F/T?} & & \text{T} & & \text{T} & \text{F} & \text{F} \\ \{ (P \rightarrow R), (Q \rightarrow S) \} \models & (P \vee Q) \rightarrow & (R \vee S) \\ \text{T/F?} & \text{F} & \text{F} & \text{F} & \text{T} & \text{F} & \text{F} & \text{F} \end{array}$$

Since we are testing entailment, we look for entailment, that is, for a truth-table row—i.e., an interpretation in the sense of an assignment of truth-values to the atomic sentence types—that makes the rhs false and makes everything on the lhs true. The rhs is a conditional, so for it to be false, the consequent (after the arrow) must be false while the antecedent (left of the arrow) is true. Since the consequent is a disjunction, for it to be false we must assign truth-values as R: F and S: F. To make the antecedent true, we need only make one or the other of “P” or “Q” true. So let’s try assigning truth-values as P: T and Q: F. This makes the whole rhs sentence false. But, since we have R: F and S: F, to make all the sentences on the lhs true, we must assign P: F and Q: F. We already have Q: F, but we were trying P: T. If we switch to P: F, that makes all the sentences on the lhs true, but now it makes the rhs sentence true as well. So there is no truth-table row that will make the lhs all true and the rhs false. Such a row would have to make “P” both T and F. That is impossible. So there is no entailment. So this is an entailment.

Problems:

- 2) $\{ (A \vee \neg B), (\neg A \rightarrow C) \} \models (\neg B \rightarrow C) ?$
- 3) $\{ ((P \vee Q) \rightarrow (R \vee S)) \} \models ((P \rightarrow Q) \wedge (R \rightarrow S)) ?$
- 4) What is the semantic status of “ $\neg(((A \vee B) \leftrightarrow (C \vee D)) \rightarrow (\neg(C \vee D) \vee (A \vee B)))$ ” ?
- 5) $(P \leftrightarrow (Q \vee R)) \models ((P \wedge Q) \vee (P \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)) ?$

IV. **Concepts:** Answer the following conceptual questions.

1. How are the semantic status of being a tautology and the status of being a logical truth related to another?
2. What is the difference between a semantic status or relation being “global” and a semantic status or relation being “particular/specific”? Which kind of relation is inconsistency?
3. What is the difference between the semantic relation of logical equivalence and the semantic relation of consistency?
4. How are the logical statuses of being a logical truth and being a logical falsehood expressed using the symbol for entailment, “ $\models$ ”? Explain why they can be expressed this way, given the definitions of these notions.

V. Mathematical Induction

1) **Example:** Here is an example of a MI proof on basic semantics (with an appeal to syntax).

Prove that a formal language (call it “SFOL*”) whose only logical operators are the sentential connectives of disjunction and conjunction (“Orandy Logic”)—governed by the usual Formation Rules and truth-tables—cannot express, for any size of input sequences of truth-values, any truth-function that is a tautology or contradiction.

Induction Set: the set of sentences of SFOL*

Partition: n = complexity (the number, n , of connectives in the sentence of SFOL*).

[So, what we want to show to hold for every case (every sentence ‘ ϕ ’ of SFOL*, at every level of complexity, n) is that each is contingent (not a logical truth, but also not a logical falsehood).]

Basis: The minimum level of cases is when ‘ ϕ ’ is atomic, i.e., $n = 0$. An atomic sentence can be either true or false, depending on which truth-value an interpretation stipulates it to have. So, ‘ ϕ ’ is contingent when $n=0$. So, the claim in question holds for the Basis.

Induction Step: The induction hypothesis (IH) for making this a proof by **strong mathematical induction** is that **all cases** (sentences) at **all** levels of complexity, $n \leq k$, are contingent. The aim of the induction step is to prove that if IH holds, then all cases at level $n=k+1$ are contingent. A sentence of SFOL*, ‘ ϕ ’, can have complexity of $k+1$ in either of two ways: i) as a conjunction of the form ‘ $(\psi \wedge \chi)$ ’, and ii) as a disjunction of the form ‘ $(\psi \vee \chi)$ ’. Since sentences of both forms have complexity level $k+1$ as a result of putting a “ $\wedge$ ” or a “ $\vee$ ” between the sentences ‘ ψ ’ and ‘ χ ’, the two sentences being combined each have complexity level $n < k$. Thus IH holds for both ‘ ψ ’ and ‘ χ ’, meaning they are both contingent sentences. Since ‘ ψ ’ and ‘ χ ’ can be either true or false, all four of the possible combinations of truth-values they can have when combined into sentences of higher complexity are given by the standard truth-table reference columns (those on the left). Since, by hypothesis, the connectives “ $\wedge$ ” and “ $\vee$ ” expresses the usual truth-functions, as displayed by the standard truth-tables for conjunction and disjunction, the following truth-tables show the possible truth-values for the two ways that a sentence, ‘ ϕ ’, of SFOL* can have complexity of $k+1$.

ψ	χ	$(\psi \wedge \chi)$	$(\psi \vee \chi)$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

As we can see from the truth-tables, both ‘ $(\psi \wedge \chi)$ ’ and ‘ $(\psi \vee \chi)$ ’ are contingent (true on some interpretation—i.e., some row in the truth-table—and false on some other: consider in particular, the interpretation in the top row that makes all the sub-sentences true, and the interpretation in the bottom row that makes all the sub-sentences false), when the sentences they combine are contingent. Since those are the only ways to form sentences of SFOL* with complexity $k+1$, all SFOL* sentences at level $k+1$ are contingent.

Ladder Principle: Reformulating the results of the Induction Step, and discharging the assumption of IH, it follows that if all cases (sentences of SFOL*) at all levels of complexity, $n \leq k$, are contingent, then all cases (sentences of SFOL*) at level $k+1$ are contingent.

Conclusion: Since we know from proving the Basis that ' ϕ ' is contingent when $n=0$, by LP it follows that ' ϕ ' is contingent when $n=1$. Since ' ϕ ' is contingent when $n=0$ **and** when $n=1$, by LP it follows that ' ϕ ' is contingent when $n=2$. And so on for all cases at all levels of complexity. Thus, all the sentences of SFOL* are contingent (meaning that, for any size of input sequences of truth-values, "Orandy Logic" cannot express any truth-function that is a tautology or contradiction).

2) **Problem:** On the model of the Example just given, construct a MI proof of the following:

Prove that a formal language (call it "SFOL#") whose only logical operators are the sentential connectives of conditional (" $\rightarrow$ ") and conjunction (" $\wedge$ ") ("Ifandy Logic")—governed by the usual Formation Rules and truth-tables—cannot express, for any size of input sequences of truth-values, any truth-function that is a contradiction. [So, what you want to show to hold for every case (every sentence ' ϕ ' of SFOL#) is that each is either contingent or a logical truth, but never a logical falsehood—that a truth-table for ' ϕ ' will have at least one T in its final column.]