

More on Multiple Quantifiers

PHIL 422

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Woodbridge

Multiple quantifiers are more complicated because we need to deal with them one at a time, in sequence, as we analyze sentences; yet, they often work together in forming the recipe for constructing interpretations. The analysis often requires that you designate the objects over which the quantifiers range by an arbitrary name (via using metalanguage variables for FOL names). Since this is a name, the resulting formula may have a truth-value, which would allow you to continue your process of decomposing compound wffs into its atomic sentences.

However, once you have completed the processing task, you need to pay attention to the way in which the objects from the domain were introduced in the proof. If, for example, a proof includes as part of the recipe, “every $\alpha \in D$,” then when you construct your interpretation, you need to make sure that “ α ” takes every value you have the interpretation put in D . As with all things, this is better understood through examples.

[I'll place explanatory commentary in red. The commentary isn't part of the sample proofs.]

A. Show that “ $\forall x \exists y (F(x) \rightarrow G(y))$ ” is logically contingent.

This proof requires that we construct two separate interpretations, one, I_t , that makes the sentence true and another, I_f , that makes the sentence false.

Proof: If “ $\forall x \exists y (F(x) \rightarrow G(y))$ ” is contingent, then some I_t makes it true, and some I_f makes it false.

Part I. On I_t :

1. “ $\forall x \exists y (F(x) \rightarrow G(y))$ ” is true.
2. From 1., every $\alpha \in D$ satisfies “ $\exists y (F(x) \rightarrow G(y))$ ” [TC $\forall$].

Since every α works here and we need to reduce this existential to something without quantifiers, we can appeal to the fact that any interpretation will assign names to objects. To do this, we simply let I_t designate the object by some arbitrarily chosen name. We indicate that this is just a placeholder name by using the metalanguage variables “ m ” and “ n ” [notice the use of double quotes here, since I am talking about the actual symbols themselves, not using them in a proof], which range over FOL names. Remember in your proof to put *single* quotes around the name-variables, since you will be talking about names (in indicating what they designate), but these are still (metalanguage) variables that take object-language names as substitution instances.

3. From 2., **for every** $\alpha \in D$, if ‘ m ’ designates α on I_t , then ‘ $\exists y (F(m) \rightarrow G(y))$ ’ is true on I_t . [Def Sat]

Notice that we are keeping explicit track of the unrestricted introduction of α , even as we consider I_t assigning an arbitrary name to α .

4. From 3., for every $\alpha \in D$, if ‘ m ’ designates α on I_t , then **some** $\beta \in D$ satisfies ‘ $(F(m) \rightarrow G(y))$ ’ on I_t . [TC $\exists$]

Notice that we still keep explicit track of how α was introduced, and of the “arbitrary name” we consider to be designating α on I_t , even as we make a restricted introduction of a new β —which we will also start keeping explicit track of. Note, although we have eliminated all the quantifiers from what we are considering, the

resulting wff isn't yet an atomic sentence (since it has an unbound FOL variable "y"); so we have more processing to do to get to that stage.

5. From 4., for every $\alpha \in D$, **there is some $\beta \in D$** , such that if ' $\mathfrak{m}$ ' designates α on I_t and ' $\mathfrak{n}$ ' designates β on I_t , then ' $(F(\mathfrak{m}) \rightarrow G(\mathfrak{n}))$ ' is true on I_t . [Def Sat]

Note that we continue to mention the original arbitrary name we are considering along with the new arbitrary name we bring in, and we now group the object introductions together (same for the name introductions).

6. From 5., for every $\alpha \in D$, there is some $\beta \in D$, such that if ' $\mathfrak{m}$ ' designates α on I_t and ' $\mathfrak{n}$ ' designates β on I_t , then either ' $F(\mathfrak{m})$ ' is false on I_t OR ' $G(\mathfrak{n})$ ' is true on I_t . [TC $\rightarrow$]

While we still need to keep track of both object introductions and both name introductions, since the internal sentential connective is no longer in the scope of any quantifiers and is not connecting any open wffs, only sentences, we can finally apply a rule for that connective and get to atomic sentences.

7. From 6., for every $\alpha \in D$, there is some $\beta \in D$, either $\alpha \notin "F"$ on I_t OR $\beta \in "G"$ on I_t . [TC Atomic]

Now that we are only talking about objects and their being in or out of predicate extensions, we can drop mention of the arbitrary names. Notice that although this is setup for two cases because it is an "OR" claim, we can actually focus on just one (or the other) of the options in constructing an I_t . So, let's focus on the first "option": for every $\alpha \in D$, $\alpha \notin "F"$. Remember, since α is unrestricted, when we construct our interpretation, we need to make **every** member of the domain **not** in the extension of "F". But the extension of "G" doesn't matter.

For I_t we can set $D = \{1\}$, "F": $\{\}$, "G": $\{\}$.

Equally, we could let I_t be a different interpretation that differs only in that it sets "G": $\{1\}$ instead. Either I will make the sentence true via focus on the first "option". [Alternatively, if we had focused on the other "option"—for some $\beta \in D$, $\beta \in "G"$ —then we could construct our I_t just by making sure we put **at least one** object from D in the extension of "G". On this focus the extension of "F" would not matter. So we could have I_t set $D = \{1\}$, "G": $\{1\}$, "F": $\{\}$ (again), or we could set $D = \{1\}$, "G": $\{1\}$, "F": $\{1\}$. So, really there are at least 3 different interpretations that work for I_t .]

Part II. On I_f :

1. " $\forall x \exists y (F(x) \rightarrow G(y))$ " is false.
2. From 1., **some $\alpha \in D$** does not satisfy " $\exists y (F(x) \rightarrow G(y))$ " on I_f . [TC $\forall$]
Again, we are trying to eliminate the quantifiers from our wffs to allow us to get to atomic sentences. Hence, we will use the definition of satisfaction to bring in a name, but we will also continue to keep track of the restricted introduction of α .
3. From 2., **for some $\alpha \in D$** , if ' $\mathfrak{m}$ ' designates α on I_f , then ' $\exists y (F(\mathfrak{m}) \rightarrow G(y))$ ' is false on I_f . [Def Sat]
4. From 3., for some $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I_f , then **every $\beta \in D$** does not satisfy ' $(F(\mathfrak{m}) \rightarrow G(y))$ ' on I_f . [TC $\exists$]

Again, we still keep track of the restricted introduction of α , and of the "arbitrary name" we consider to designate α on I , even as we make an unrestricted introduction of β —which we will also start keeping explicit track of. And we are again trying to

eliminate all the quantifiers from our wffs to allow us to get to an atomic sentence.
So, we will use the definition of satisfaction to bring in a name.

5. From 4., there is some $\alpha \in D$, such that **for every $\beta \in D$** , if ' $\mathfrak{m}$ ' designates α on I_f and ' $\mathfrak{n}$ ' designates β on I_f , then ' $(F(\mathfrak{m}) \rightarrow G(\mathfrak{n}))$ ' is false on I_f . [Def Sat]
6. From 5., there is some $\alpha \in D$, such that for every $\beta \in D$, if ' $\mathfrak{m}$ ' designates α on I_f and ' $\mathfrak{n}$ ' designates β on I_f , then ' $F(\mathfrak{m})$ ' is true on I_f AND ' $G(\mathfrak{n})$ ' is true β on I_f . [TC $\rightarrow$]
7. From 6., there is some $\alpha \in D$, such that for every $\beta \in D$, $\alpha \in "F"$ on I_f [TC Atomic] AND $\beta \notin "G"$ on I_f . [TC Atomic]

We can now drop tracking the name introductions, since we are just considering objects being in and out of the extensions of predicates. We now have a recipe for the interpretation. As an "AND" claim, it involves just one case.

For I_f set $D = \{1\}$, " F ": $\{1\}$, " G ": $\{\}$.

We have constructed an interpretation, I_t , that makes the sentence true, and another interpretation, I_f , that makes the sentence false. [This kind of restatement of the results isn't strictly required in proofs; however, even the tersest mathematical proofs often restate the major results, as it is so helpful in reading the proof.] Hence, the sentence is logically contingent (which is what we needed to show). [This statement connects the major result with the relevant definition.]

B. Show " $(\forall x \exists y R(x,y) \rightarrow \exists x \exists y R(x,y))$ " is logically true.

Proof: S'pose not, i.e., suppose there is an interpretation, I , that makes the sentence false.
So, I makes

1. " $\forall x \exists y R(x,y)$ " true AND
2. " $\exists x \exists y R(x,y)$ " false. [TC $\rightarrow$]

Since " $\rightarrow$ " is the main connective, not in the scope of any quantifiers, we process it first. Notice that the result is a conjunction of claims, so both will need to be jointly considered for the purposes of interpretation later. For now, we will consider each conjunct separately.

3. From 1., **every $\alpha \in D$** satisfies " $\exists y R(x,y)$ " on I . [TC $\forall$]
4. From 3., **for every $\alpha \in D$** , if ' $\mathfrak{m}$ ' designates α on I , then ' $\exists y R(\mathfrak{m},y)$ ' is true on I . [Def Sat]
5. From 4., for every $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I , then **some $\beta \in D$** satisfies ' $R(\mathfrak{m},y)$ ' on I . [TC $\exists$]
6. From 5., for every $\alpha \in D$, **there is some $\beta \in D$** , such that if ' $\mathfrak{m}$ ' designates α on I and ' $\mathfrak{n}$ ' designates β on I , then ' $R(\mathfrak{m},\mathfrak{n})$ ' is true on I . [Def Sat]
7. From 6., for every $\alpha \in D$, there is a $\beta \in D$, such that the ordered pair, $\langle \alpha, \beta \rangle \in "R"$ on I . [TC Atomic]
8. From 2., **every $\alpha \in D$** does not satisfy " $\exists y R(x,y)$ " on I . [TC $\exists$]
Since we want to cover every object in the domain again, we can consider this in terms of α again, since α was introduced previously as unrestricted.
9. From 8., **for every $\alpha \in D$** , if ' $\mathfrak{m}$ ' designates α on I , then ' $\exists y R(\mathfrak{m},y)$ ' is false on I . [Def Sat]

10. From 9., for every $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then **every $\gamma \in D$** does not satisfy ' $R(\mathfrak{m}, \gamma)$ ' on I. [TC $\exists$]
Even though the new object variable introduced is also unrestricted, it will help with clarity to use a new variable here.
11. From 10., for every $\alpha \in D$ and every $\gamma \in D$, if ' $\mathfrak{m}$ ' designates α on I **and ' $\mathfrak{n}$ ' designates γ on I**, then ' $R(\mathfrak{m}, \mathfrak{n})$ ' is false on I. [Def Sat]
12. From 11., for every $\alpha \in D$ and every $\gamma \in D$, $\langle \alpha, \gamma \rangle \notin "R"$ on I. [TC Atomic]
 From 7., α covers all members of the domain, but, for each α , β is a particular member of the domain. And in 12., α again covers all the members of the domain (this α is the same as the prior α). But in 12., for each α , we have γ covering **all** the members of the domain. This, of course, includes, in each case, the particular object β paired with α in 7. So, for each α in 12., consider when γ is the β paired with that α in 7. But then we have both that for every $\alpha \in D$, there is some $\beta \in D$ such that $\langle \alpha, \beta \rangle \in "R"$ on I (by 7.) and that for every $\alpha \in D$ and that $\beta \in D$, $\langle \alpha, \beta \rangle \notin "R"$ on I (by 12., via substitution of β for γ). But this is a contradiction. Hence, no interpretation can make the sentence false. Thus, the sentence is logically true.

So, with the more careful procedures just explained, consider the problem that I got off track explaining in the lecture today:

C. Show $\{ \forall x \exists y R(x, y) \} \not\models \exists y \forall x R(x, y)$

Proof: If there is "enfailment", then some I makes

1. " $\forall x \exists y R(x, y)$ " true AND
2. " $\exists y \forall x R(x, y)$ " false.
3. From 1., **every $\alpha \in D$** satisfies " $\exists y R(x, y)$ " on I. [TC $\forall$].
4. From 3., **for every $\alpha \in D$** , if ' $\mathfrak{m}$ ' designates α on I, then ' $\exists y R(\mathfrak{m}, y)$ ' is true on I. [Def Sat]
5. From 4., for every $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then **some $\beta \in D$** satisfies ' $R(\mathfrak{m}, y)$ ' on I. [TC $\exists$]
6. From 5., for every $\alpha \in D$, **there is some $\beta \in D$** , such that if ' $\mathfrak{m}$ ' designates α on I **and ' $\mathfrak{n}$ ' designates β on I**, then ' $R(\mathfrak{m}, \mathfrak{n})$ ' is true on I. [Def. Sat.]
7. From 6., for every $\alpha \in D$, there is some $\beta \in D$, such that $\langle \alpha, \beta \rangle \in "R"$ on I. [TC Atomic]
8. From 2., **every $\alpha \in D$** does not satisfy " $\forall x R(x, y)$ " on I. [TC $\exists$]
9. From 8., for every $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then ' $\forall x R(x, \mathfrak{m})$ ' is false on I. [Def. Sat.] **[Note that here the first name goes in the second position in the string.]**
10. From 9., for every $\alpha \in D$, if ' $\mathfrak{m}$ ' designates α on I, then **some $\gamma \in D$** does not satisfy ' $R(x, \mathfrak{m})$ ' on I. [TC $\forall$]
11. From 10., for every $\alpha \in D$, there is some $\gamma \in D$, such that if ' $\mathfrak{m}$ ' designates α on I **and ' $\mathfrak{n}$ ' designates γ on I**, then ' $R(\mathfrak{n}, \mathfrak{m})$ ' is false on I. [Def. Sat.] **[Note that the second name goes in the first position in the string.]**
12. From 11., for every $\alpha \in D$, there is some $\gamma \in D$, such that $\langle \gamma, \alpha \rangle \notin "R"$ on I. [TC Atomic] **[Given the order of the names in the string, this means that γ comes before α in the ordered pair, even though γ was introduced after α and is relativized to α .]**

Note that 5. and 10. involve two different restricted introductions of objects. So the domain, D , of the I we construct based on the recipe generated by this processing must contain at least two objects. So, set $D = \{1, 2\}$. Then the ordered pairs possible from D are $\langle 1, 1 \rangle$, $\langle 1, 2 \rangle$, $\langle 2, 1 \rangle$, and $\langle 2, 2 \rangle$. By 7., every object in D must be first in an ordered pair with some object in D second, where I puts that ordered pair in the extension of “ R ”. So, either $\langle 1, 1 \rangle \in \text{“}R\text{”}$ on I or $\langle 1, 2 \rangle \in \text{“}R\text{”}$ on I AND either $\langle 2, 1 \rangle \in \text{“}R\text{”}$ on I or $\langle 2, 2 \rangle \in \text{“}R\text{”}$ on I . By 12., every object in D must be second in an ordered pair with some object in D first, where I excludes that ordered pair from the extension of “ R ”. So, either $\langle 1, 1 \rangle \notin \text{“}R\text{”}$ on I or $\langle 2, 1 \rangle \notin \text{“}R\text{”}$ on I AND either $\langle 1, 2 \rangle \notin \text{“}R\text{”}$ on I or $\langle 2, 2 \rangle \notin \text{“}R\text{”}$ on I . So, as mentioned in lecture, we can let I be as follows: $D = \{1, 2\}$, “ R ”: $\{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$. This I makes the sentence in the set on the lhs true and makes the sentence on the rhs false, thus demonstrating entailment.