

EXAMPLES OF MATHEMATICAL INDUCTION ON SYNTAX

1. Use **mathematical induction** to prove that the number of sub-wffs of any wff of SFOL (the sentential fragment of FOL) with n connectives is less than or equal to $2n + 1$. Take 'Sub(ϕ)' to stand for the set of sub-wffs of a wff ' ϕ ', and ' $|\text{Sub}(\phi)|$ ' to stand for the number of members in that set, i.e., the number of sub-wffs that occur in a wff ' ϕ '. So what you want to prove is that if ' ϕ ' has n connectives, then $|\text{Sub}(\phi)| \leq 2n + 1$.

Induction Set: the wffs of SFOL

Partition: complexity (in terms of the number n of connective tokens a wff contains)

Basis: The minimal level of complexity is $n = 0$: the atomic sentences/wffs of SFOL. The only sub-wff an atomic wff has is itself. So, $\text{Sub}(\phi) = \{\phi\}$ when $n = 0$, meaning that $|\text{Sub}(\phi)| = 1$, but since $1 \leq 2(0) + 1$, the Basis checks out.

Induction Step: Assume (as IH): $|\text{Sub}(\phi)| \leq 2n + 1$ for all cases **at all levels** $n \leq k$, for some arbitrary unspecified level k . [Note: this stronger assumption makes this a case of **strong MI**.] Now prove that $|\text{Sub}(\phi)| \leq 2n + 1$ for all cases at level $n = k+1$.

There are 5 ways that a wff ' ϕ ' can be a $k+1$ -level complexity wff of SFOL: as i) a negation, ii) a conjunction, iii) a disjunction, iv) a conditional, or v) a biconditional. Since ways ii)-v) all have the same binary-connective structure, they can be dealt with all together. But i) is different and requires its own proof. If ' ϕ ' is a $k+1$ -level negation, then ' ϕ ' is formed by adding a negation sign, " $\neg$ ", in front of some wff ' ψ ' of SFOL of complexity level $n = k$. Since ' ψ ' is at level k , by the IH, $|\text{Sub}(\psi)| \leq 2k + 1$. Since ' ϕ ' = ' $\neg\psi$ ', we know that the set $\text{Sub}(\phi)$ just is the set $\text{Sub}(\psi)$ with one more sub-wff added, namely ' ϕ '/' $\neg\psi$ ' itself. Thus, $|\text{Sub}(\phi)| = |\text{Sub}(\psi)| + 1$. Since $|\text{Sub}(\psi)| \leq 2k + 1$, that means $|\text{Sub}(\psi)| + 1 \leq 2k+1+1$. Thus, $|\text{Sub}(\phi)| \leq 2k+1+1$ or $2k+2$, i.e., $2(k+1)$, which is less than $2(k+1) + 1$. So, $|\text{Sub}(\phi)| \leq 2(k+1) + 1$ when ' ϕ ' is a $k+1$ -level negation.

For ways ii)-v) that ' ϕ ' can be a $k+1$ -level wff of SFOL, notice that ' ϕ ' will for each of these ways be formed from two wffs of SFOL, ' ψ ' and ' χ ', by placing a binary connective between them, inside parentheses. Let "*" be a metalanguage variable for binary connectives of SFOL. Then ways ii)-v) that ' ϕ ' can be a $k+1$ -level wff of SFOL will all have the form ' $(\psi * \chi)$ '. Since ' ψ ' and ' χ ' each have some (possibly but not necessarily the same) complexity level, neither of which will be level k , we need to do **strong MI** here. So, say that ' ψ ' has complexity level i and ' χ ' has complexity level j . Since our $k+1$ -level ' ϕ ' is formed by putting one more connective in a string combining ' ψ ' and ' χ ' to get to level $k+1$, that means that $(i+j)=k$. Thus, both i and j are $\leq k$, meaning IH holds for both ' ψ ' and ' χ '. So, $|\text{Sub}(\psi)| \leq 2i + 1$ and $|\text{Sub}(\chi)| \leq 2j + 1$. We also know that $\text{Sub}((\psi * \chi))$ adds only one more sub-wff to $\text{Sub}(\psi) \cup \text{Sub}(\chi)$, namely ' $(\psi * \chi)$ ' itself. So, since ' ϕ ' = ' $(\psi * \chi)$ ', $|\text{Sub}(\phi)| = |\text{Sub}(\psi)| + |\text{Sub}(\chi)| + 1$. But since $|\text{Sub}(\psi)| \leq 2i + 1$ and $|\text{Sub}(\chi)| \leq 2j + 1$, that means that $|\text{Sub}(\phi)| \leq (2i+1) + (2j+1) + 1$, which is just $2(i+j+1) + 1$,

which is just $2(k+1) + 1$. So, if ‘ ϕ ’ is a $k+1$ -level wff of SFOL in any of ways ii)-v), then $|\text{Sub}(\phi)| \leq 2(k+1) + 1$. So this holds for all cases at level $k+1$.

Ladder Principle: Reformulating the results of the Induction Step, by discharging the assumption (of IH), we get the following conditional: If for all cases at all levels $n \leq k$, $|\text{Sub}(\phi)| \leq 2n + 1$, then for all cases at level $n=k+1$, $|\text{Sub}(\phi)| \leq 2n + 1$.

Conclusion: We proved in the Basis that $|\text{Sub}(\phi)| \leq 2n + 1$ holds for all cases at the minimal level of $n=0$. By the “Ladder Principle” (LP), this now gives us that $|\text{Sub}(\phi)| \leq 2n + 1$ also holds for all cases at level $n=1$. Since we now have that $|\text{Sub}(\phi)| \leq 2n + 1$ holds for all cases both at level $n=0$ and at level $n=1$, the **[strong MI]** LP now gives us that $|\text{Sub}(\phi)| \leq 2n + 1$ holds for all cases at level $n=2$. [Just to make the strong MI clear here: Since we now have proved that $|\text{Sub}(\phi)| \leq 2n + 1$ holds for all cases at all of levels $n=0, 1, 2$, the LP now gives us that $|\text{Sub}(\phi)| \leq 2n + 1$ holds for all cases at level $n=3$.] And so on, for all cases at all levels. So, in general, the number of sub-wffs of any wff of SFOL with n connectives is less than or equal to $2n + 1$.

2. Use **mathematical induction** to prove that for any wff of FOL, the number of atomic sentence tokens in the wff is the number of binary operator tokens in the wff, plus one. For any wff ‘ ϕ ’ of FOL, let “ $|\phi|_a$ ” stand for the number of atomic sentence tokens in ‘ ϕ ’, and let “ $|\phi|_b$ ” stand for the number of binary operator tokens in ‘ ϕ ’. Let “ $|\phi|_o$ ” stand for the number of operator tokens (including both binary and unary operators) in ‘ ϕ ’. Let “ $@$ ” be a metalanguage variable for any unary operator, i.e., “ $\neg$ ” or “ $\exists$ ” or “ $\forall$ ”. Let “ $*$ ” be a metalanguage variable for any binary operator, i.e., “ $\wedge$ ” or “ $\vee$ ” or “ $\rightarrow$ ” or “ $\leftrightarrow$ ”. (So, show that for any ‘ ϕ ’, $|\phi|_a = |\phi|_b + 1$.)

Induction Set: the wffs of FOL

Partition: complexity (in terms of the number n of operator tokens a wff contains, so $|\phi|_o = n$)

Basis: The minimal level is $|\phi|_o = 0$, i.e., when ‘ ϕ ’ is atomic. Then $|\phi|_a = 1$ and $|\phi|_b = 0$. So, when $|\phi|_o = 0$, $|\phi|_a = |\phi|_b + 1$. So, the Basis is proven.

Inductive Hypothesis: For all cases at all levels $n = |\phi|_o \leq k$, let $|\phi|_a = |\phi|_b + 1$. We need to show that for all cases at level $n = |\phi|_o = (k + 1)$, $|\phi|_a = |\phi|_b + 1$.

There are two kinds of cases. In one (kind of) case, the main operator of ‘ ϕ ’ is unary. In another (kind of) case, the main operator of ‘ ϕ ’ is binary.

Case 1: $|\phi|_o = k + 1$ and ‘ ϕ ’ is ‘ $@\psi$ ’ for some unary operator ‘ $@$ ’ and wff ‘ ψ ’. We need to show that $|\phi|_a = |\psi|_b + 1$. Since $|\phi|_o = k + 1$, $|\psi|_o = k$. So, $|\psi|_a = |\psi|_b + 1$ (by IH). Moreover, since the main operator of ‘ ϕ ’ is ‘ $@$ ’, $|\phi|_a = |\psi|_a$, since unary operations do not add atoms. Also, $|\phi|_b = |\psi|_b$ because unary operations do not add binary operations. Making substitutions based on these identities into the IH regarding ‘ ψ ’, this gives: $|\psi|_a = |\psi|_b + 1$, which is what we wanted to prove.

Case 2: $|\phi|_o = k + 1$, and ‘ ϕ ’ is ‘ $(\psi * \chi)$ ’ for some binary operation ‘ $*$ ’ and wffs ‘ ψ ’ and ‘ χ ’. We want to show that $|(\psi * \chi)|_a = |(\psi * \chi)|_b + 1$. Since $|(\psi * \chi)|_o = k + 1$, both $|\psi|_o \leq k$ and $|\chi|_o \leq k$. Hence (by IH), ① $|\psi|_a = |\psi|_b + 1$, and ② $|\chi|_a = |\chi|_b + 1$. Note well that ③ $|(\psi * \chi)|_a = |\psi|_a + |\chi|_a$, since the operator ‘ $*$ ’ combines the two wffs without adding any more wffs, and ④ $|(\psi * \chi)|_b = |\psi|_b + |\chi|_b + 1$, because ‘ ϕ ’ includes all of the binary operations in ‘ ψ ’ and the binary operations in ‘ χ ’ PLUS the new operation ‘ $*$ ’. Let’s start with $|\psi|_a$ and add to it $|\chi|_a$, i.e., $|\psi|_a + |\chi|_a = ?$ Given ① and ②, $|\psi|_a + |\chi|_a = |\psi|_b + 1 + |\chi|_b + 1$ [call this ⑤]. Rearrange the terms of ⑤ to: $|\psi|_a + |\chi|_a = |\psi|_b + |\chi|_b + 1 + 1$. Now in the rearranged ⑤, replace $|\psi|_a + |\chi|_a$ with $|(\psi * \chi)|_a$ because of ③ and replace $|\psi|_b + |\chi|_b + 1$ with $|(\psi * \chi)|_b$ because of ④: $|(\psi * \chi)|_a = |(\psi * \chi)|_b + 1$. And this is what we wanted to show.

Ladder Principle: Since both cases in the Induction Step lead to $|\phi|_a = |\phi|_b + 1$ for $|\phi|_o = k + 1$ when we assume IH, we discharge the assumption and conclude: If for all cases at all levels $|\phi|_o \leq k$ we have $|\phi|_a = |\phi|_b + 1$, then, for all cases at $|\phi|_o = k + 1$, $|\phi|_a = |\phi|_b + 1$.

Conclusion: Since we proved in the Basis that for all cases at level $|\phi|_o = 0$, $|\phi|_a = |\phi|_b + 1$, the LP gets us that for cases at level $|\phi|_o = 1$, $|\phi|_a = |\phi|_b + 1$. Since we have this result for all cases at level 0 **and** all cases at level 1, the LP gets us that for all cases at level $|\phi|_o = 2$, $|\phi|_a = |\phi|_b + 1$. [Since we have this result for all cases at level 0 **and** all cases at level 1 **and** all cases at level 2, the LP now gets us that for all cases at level $|\phi|_o = 3$, $|\phi|_a = |\phi|_b + 1$]. And so on, for all cases at all levels. So, in general, for any wff of FOL, the number of atomic sentence tokens in the wff is the number of binary operator tokens in the wff, plus one.