INTRODUCTION TO SYMBOLIC LOGIC
PHIL 114, Sec. 1001
University of Nevada, Las Vegas

Extra Credit: Derivation Trees

1. The Tree Rules

We have nine tree rules for our sentential (truth-functional) connectives. (We'll leave aside the tree rules for the quantifiers and the identity-predicate for present purposes.)

Double Negation (DN):

$$\begin{array}{c} \neg\neg\phi \\ | \\ \phi \end{array}$$

Conjunction:

$$\begin{array}{c} (\phi \wedge \psi) \\ | \\ \phi \\ \psi \end{array}$$

Negated Conjunction:

$$\begin{array}{ccc} & \neg(\phi \wedge \psi) & \\ \diagup & & \diagdown \\ \neg\phi & & \neg\psi \end{array}$$

Disjunction:

$$\begin{array}{ccc} & (\phi \vee \psi) & \\ \diagup & & \diagdown \\ \phi & & \psi \end{array}$$

Negated Disjunction:

$$\begin{array}{c} \neg(\phi \vee \psi) \\ | \\ \neg\phi \\ \neg\psi \end{array}$$

Conditional:

$$\begin{array}{ccc} & (\phi \rightarrow \psi) & \\ \diagup & & \diagdown \\ \neg\phi & & \psi \end{array}$$

Negated Conditional:

$$\begin{array}{c} \neg(\phi \rightarrow \psi) \\ | \\ \phi \\ \neg\psi \end{array}$$

Biconditional:

$$\begin{array}{ccc} & (\phi \leftrightarrow \psi) & \\ \diagup & & \diagdown \\ \phi & & \neg\phi \\ \psi & & \neg\psi \end{array}$$

Negated Biconditional:

$$\begin{array}{ccc} & \neg(\phi \leftrightarrow \psi) & \\ \diagup & & \diagdown \\ \phi & & \neg\phi \\ \neg\psi & & \psi \end{array}$$

We *decompose* a wff or set of wffs by first listing the wffs vertically. We call this the *trunk*.
Then, for each non-literal wff, apply the appropriate rule to decompose it into its simpler parts.
When you apply a rule to a wff, mark it with "✓".
Write the result of applying the rule at the bottom of *every* open branch below the wff you decompose. Never "jump across" branches or go back "upstream" to get to an open node.
A branch *closes* if it contains some sentence, ϕ, and its negation, $\neg\phi$.
We mark a closed branch with "⊗".
A branch is complete if it is either closed OR all of its non-literal wffs have been checked.
If a branch is complete and not closed, we call it open (really, we should call it a "completed open branch"). We mark completed open branches with "○".
A tree is closed if all of its branches are closed.
A tree is open if at least one of its branches is a completed open branch.

2. Theorems, Anti-theorems, and Neutrals

We can test whether a wff is a theorem, an anti-theorem or a neutral by using trees. To do so, first, place the wff in the trunk. Then, decompose until all branches are completed. Check for closed branches. Then, put the negation of the wff in the trunk of a new tree. Next, decompose until all branches are completed. Finally, check for closed branches.

Definitions:

- A wff is a *theorem* iff its tree is open and its negation's tree is closed.
- A wff is an *anti-theorem* iff its tree is closed and its negation's tree is open.
- A wff is a *neutral* iff both its tree and its negation's tree are open.

3. Compatible and Incompatible Sets

We can test whether a set of wffs is compatible using trees. To do so, first, place all of the members of the set in the trunk. Then decompose the trunk until all the branches are complete. Check for closed branches.

Definitions:

- A set of wffs is *compatible* iff its completed tree is open.
- A set of wffs is *incompatible* iff its completed tree is closed.

4. Establishments and Non-establishments

Definitions:

- A set of wffs, Γ, *establishes* a wff, ϕ, iff the set of wffs $\Sigma = \Gamma \cup \{\neg\phi\}$ is incompatible.
- A set of wffs, Γ, *non-establishes* a wff, ϕ, iff the set of wffs $\Sigma = \Gamma \cup \{\neg\phi\}$ is compatible.

Note: We write 'Γ establishes ϕ' using the "single turnstile", as in '$\Gamma \vdash \phi$'.

5. Coupled and Uncoupled Pairs

Definitions:

- A pair of wffs, ϕ and ψ, are *coupled* iff $\phi \vdash \psi$ and $\psi \vdash \phi$.
- A pair of wffs, ϕ and ψ, are *uncoupled* iff either $\phi \vdash \psi$ fails or $\psi \vdash \phi$ fails.

6. Problems: Answer the following questions by constructing the relevant trees.

1. Is $((P(a) \vee Q(b)) \leftrightarrow \neg(\neg Q(b) \wedge \neg P(a)))$ a theorem, anti-theorem, or neutral?
2. Is $\{ (P(a) \rightarrow Q(a)), (Q(a) \leftrightarrow R(a)), (\neg R(a) \rightarrow \neg P(a)) \}$ compatible or incompatible?
3. Is $\{ \neg(P(a) \rightarrow Q(b)), \neg(P(a) \vee Q(a)), (R(a) \rightarrow Q(b)), \neg Q(b) \}$ compatible or incompatible?
4. Does $(P(a) \wedge Q(b)), (P(a) \rightarrow R(a)), \neg R(a) \vdash S(b)$?
5. Does $\neg(P(a) \leftrightarrow Q(a)) \vdash (P(a) \rightarrow \neg Q(a))$?
6. Are $(P(a) \rightarrow Q(a))$ and $\neg(P(a) \wedge \neg Q(a))$ coupled our uncoupled?